3.2. Let \( y = e^{\lambda t} \) in \( y'' + 5y' + 6y = 0 \) to obtain
\[
\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} + 6e^{\lambda t} = 0,
\]
\[
e^{\lambda t}(\lambda^2 + 5\lambda + 6) = 0.
\]
Because \( e^{\lambda t} \neq 0 \), we arrive at the characteristic equation
\[
\lambda^2 + 5\lambda + 6 = 0,
\]
\[
(\lambda + 3)(\lambda + 2) = 0.
\]

3.6. Let \( y = e^{\lambda t} \) in \( 6y'' + 5y' - 5y = 0 \) to obtain
\[
6\lambda^2 e^{\lambda t} + 5\lambda e^{\lambda t} - 6e^{\lambda t} = 0,
\]
\[
e^{\lambda t}(6\lambda^2 + 5\lambda - 6) = 0.
\]
Because \( e^{\lambda t} \neq 0 \), we arrive at the logistic equation
\[
6\lambda^2 + 5\lambda - 6 = 0
\]
\[
(3\lambda - 2)(2\lambda + 3) = 0.
\]

3.10. If \( y'' + 2y' + 17y = 0 \), then the characteristic equation is
\[
\lambda^2 + 2\lambda + 17 = 0.
\]
The roots of the characteristic equation are \(-1 \pm 4i\), leading to the complex solutions
\[
z(t) = e^{-(1+4i)t} \quad \text{and} \quad \bar{z}(t) = e^{-(1-4i)t}.
\]
However, by Euler's identity,
\[
z(t) = e^{-t}e^{4it} = e^{-t} (\cos 4t + i \sin 4t),
\]
and the real and imaginary parts of \( z \) lead to a fundamental set of real solutions \( y_1(t) = e^{-t} \cos 4t \) and \( y_2(t) = e^{-t} \sin 4t \). Hence the general solution is
\[
y(t) = C_1 e^{-t} \cos 4t + C_2 e^{-t} \sin 4t.
\]

3.14. If \( y'' - 6y' + 9y = 0 \), then the characteristic equation is
\[
\lambda^2 - 6\lambda + 9 = (\lambda - 3)^2 = 0.
\]
Hence the characteristic equation has a repeated root, \( \lambda = 3 \). Therefore, \( y_1(t) = e^{3t} \) and \( y_2(t) = te^{3t} \) form a fundamental set of real solutions. Hence, the general solution is
\[
y(t) = C_1 e^{3t} + C_2 te^{3t} = (C_1 + C_2 t)e^{3t}.
\]

3.20. If \( y'' - 2y' + 17y = 0 \), then the characteristic equation is
\[
\lambda^2 - 2\lambda + 17 = 0.
\]
with roots \( 1 \pm 4i \). The complex solution,
\[
z(t) = e^{(1+4i)t} = e^t (\cos 4t + i \sin 4t)
\]
leads to a fundamental set of real solutions and the general solution
\[
y(t) = e^t (C_1 \cos 4t + C_2 \sin 4t).
\]
The initial condition \( y(0) = -2 \) provides
\[
-2 = C_1.
\]
Differentiating the general solution,
\[
y'(t) = e^t (C_1 \cos 4t + C_2 \sin 4t) + e^t (-4C_1 \sin 4t + 4C_2 \cos 4t),
\]
3.22. If $y'' + 10y' + 25y = 0$, then the characteristic equation is

$$\lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 = 0,$$

with repeated root $\lambda = -5$. This leads to the fundamental set of solutions $y_1(t) = e^{-5t}$ and $y_2(t) = te^{-5t}$ and the general solution is

$$y(t) = C_1e^{-5t} + C_2te^{-5t} = (C_1 + C_2t)e^{-5t}.$$

Using the initial condition $y(0) = 2$ leads to

$$2 = C_1.$$

Differentiating the general solution,

$$y'(t) = C_2e^{-5t} - 5(C_1 + C_2t)e^{-5t},$$

then the initial condition $y'(0) = -1$ leads to

$$-1 = C_2 - 5C_1.$$

Thus, $C_1 = 2$ and $C_2 = 9$ and the final solutions is

$$y(t) = (2 + 9t)e^{-5t}.$$

---

4.4 Harmonic Motion

4.11. Substitute $m = 0.2$ kg and $k = 5$ kg/s$^2$ in $my'' + ky = 0$ to obtain $0.2y'' + 5y = 0$ or

$$y'' + 25y = 0.$$

The characteristic equation is $\lambda^2 + 25 = 0$, with zeros $\lambda = \pm 5i$, so

$$z(t) = e^{5it} = \cos 5t + i \sin 5t$$

is a complex solution. The real and imaginary parts of this solution form a fundamental set of real solutions, giving the general solution

$$y(t) = C_1 \cos 5t + C_2 \sin 5t.$$

The initial displacement is 0.5 m, so $y(0) = 0.5$ and $C_1 = 0.5$. Differentiating,

$$y'(t) = -5C_1 \sin 5t + 5C_2 \cos 5t.$$

The system is released from rest, so $y'(0) = 0$ and $C_2 = 0$. Thus, the solution is

$$y(t) = 0.5 \cos 5t,$$

which has amplitude 0.5, frequency 5 rad/s, and zero phase.
4.14. The system \( mx'' + kx = 0 \), or \( x'' + (k/m)x = 0 \), is equivalent to

\[
x'' + \omega_0^2 x = 0,
\]

with \( \omega_0^2 = k/m \). The characteristic equation is \( \lambda^2 + \omega_0^2 = 0 \), with roots \( \lambda = \pm \omega_0 i \). Thus, we have complex solution

\[
z(t) = e^{i\omega_0 t} = \cos \omega_0 t + i \sin \omega_0 t.
\]

The real and imaginary parts of this solution form a fundamental set of solutions and provide the general solution

\[
x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.
\]

The initial condition \( x(0) = x_0 \) gives \( C_1 = x_0 \). Differentiate.

\[
x'(t) = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t.
\]

The initial condition \( x'(0) = v_0 \) gives \( C_2 = v_0/\omega_0 \). Thus, the solution is

\[
x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t.
\]

Plot the coefficients, calculate the magnitude of the vector, and mark the angle.

![Diagram](image)

The tangent of the angle \( \phi \) is easily calculated.

\[
\tan \phi = \frac{v_0}{x_0 \omega_0}
\]

Factor out the magnitude as follows.

\[
x(t) = \sqrt{x_0^2 + v_0^2/\omega_0^2} \left( \frac{x_0}{\sqrt{x_0^2 + v_0^2/\omega_0^2}} \cos \omega_0 t + \frac{v_0/\omega_0}{\sqrt{x_0^2 + v_0^2/\omega_0^2}} \sin \omega_0 t \right)
\]
But \( \cos \phi = x_0 / \sqrt{x_0^2 + v_0^2 / \omega_0^2} \) and \( \sin \phi = (v_0 / \omega_0) / \sqrt{x_0^2 + v_0^2 / \omega_0^2} \), so we can write

\[
x(t) = \sqrt{x_0^2 + v_0^2 / \omega_0^2} (\cos \omega_0 t + \sin \phi \sin \omega_0 t)
\]

\[
x(t) = \sqrt{x_0^2 + v_0^2 / \omega_0^2} \cos(\omega_0 t - \phi).
\]

Thus, the amplitude of the motion is

\[
A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}
\]

\[
A = \sqrt{x_0^2 + \frac{v_0^2}{k/m}}
\]

\[
A = \sqrt{x_0^2 + \frac{mv_0^2}{k}}.
\]

4.18. By Hooke's Law,

\[
k = \frac{F}{y} = \frac{mg}{y}
\]

\[
k = \frac{(0.05 \text{ kg})(9.8 \text{ m/s}^2)}{(0.2 \text{ m})}
\]

\[
k = 2.45 \text{ N/m}.
\]

Hence, \( my'' + \mu y' + ky = 0 \) becomes \( 0.05 y'' + \mu y' + 2.45 y = 0 \), or

\[
y'' + 20 \mu y' + 49 y = 0.
\]

This system has characteristic equation \( \lambda^2 + 20 \mu \lambda + 49 = 0 \), with zeros given by

\[
\lambda = \frac{-20 \mu \pm \sqrt{400 \mu^2 - 196}}{2}.
\]

The system is critically damped if it has one single, repeated root. The happens only if

\[
400 \mu^2 - 196 = 0
\]

\[
\mu^2 = \frac{196}{400}
\]

\[
\mu = \frac{14}{20}.
\]

Thus, \( \mu = 7/10 \), and our equation becomes

\[
y'' + 14y' + 49y = 0,
\]

whose characteristic equation \( \lambda^2 + 14 \lambda + 49 = (\lambda + 7)^2 = 0 \) has a repeated root \( \lambda = -7 \). Thus, the general solution is

\[
y(t) = (C_1 + C_2 t) e^{-7t}.
\]

If we assume that the mass is displaced in a downward direction, \( y(0) = -0.15 \text{ m} \) and \( C_1 = -0.15 \). Differentiate.

\[
y'(t) = C_2 e^{-7t} - 7(C_1 + C_2 t) e^{-7t}
\]

If the mass is released from rest, then \( y'(0) = 0 \) and

\[
0 = C_2 - 7C_1.
\]
Thus, $C_2 = -1.05$ and the solution is

$$y(t) = -(0.15 + 1.05t)e^{-7t}$$

4.5 Inhomogeneous Equations; the Method of Undetermined Coefficients

5.2. Let $y(t) = Ae^{-t}$. Then

$$y'(t) = -Ae^{-t},$$

$$y''(t) = Ae^{-t},$$

and $y'' + 6y' + 8y = -3e^{-t}$ becomes

$$Ae^{-t} + 6(-Ae^{-t}) + 8(Ae^{-t}) = -3e^{-t}$$

$$3A = -3$$

$$A = -1.$$  

Thus, $y = e^{-t}$ is a particular solution.

5.4. Let $y(t) = Ae^{2t}$. Then,

$$y'(t) = 2Ae^{2t},$$

$$y''(t) = 4Ae^{2t},$$

and $y'' + 3y' - 18y = 18e^{2t}$ becomes

$$4Ae^{2t} + 3(2Ae^{2t}) - 18(Ae^{2t}) = 18e^{2t}$$

$$-8A = 18$$

$$A = -\frac{9}{4}.$$  

Thus, $y = -(9/4)e^{2t}$ is a particular solution.

5.8. Let $y_p = a \cos 3t + b \sin 3t$. Then

$$y'_p = -3a \sin 3t + 3b \cos 3t,$$

$$y''_p = -9a \cos 3t - 9b \sin 3t,$$

and the equation $y'' + 7y' + 10y = -4 \sin 3t$ becomes

$$(a + 21b) \cos 3t + (-21a + b) \sin 3t = -4 \sin 3t.$$

Thus,

$$a + 21b = 0$$

$$-21a + b = -4,$$

leading to $a = 42/221$ and $b = -2/221$ and the particular solution $y_p = (42/221) \cos 3t - (2/221) \sin 3t$.  

5
5.12. Let \( z = Ae^{2t} \). Then

\[
\begin{align*}
  z' &= 2iAe^{2t} \\
  z'' &= (2i)^2Ae^{2t},
\end{align*}
\]

and \( z'' + 7z' + 6z = 3e^{2t} \) leads to

\[
(2i)^2Ae^{2t} + 7(2i)Ae^{2t} + 6Ae^{2t} = 3e^{2t}
\]

\[
((2i)^2 + 7(2i) + 6) A = 3
\]

\[
A = \frac{3}{2 + 14i}
\]

\[
A = \frac{3}{100} - \frac{21}{100}i.
\]

Thus,

\[
z = \left( \frac{3}{100} - \frac{21}{100}i \right) e^{2t}
\]

\[
z = \left( \frac{3}{100} - \frac{21}{100}i \right)(\cos 2t + i \sin 2t)
\]

\[
z = \left( \frac{3}{100} \cos 2t + \frac{21}{100} \sin 2t \right) + i\left( -\frac{21}{100} \cos 2t + \frac{3}{100} \sin 2t \right)
\]

is a solution of \( z'' + 7z' + 6z = 3e^{2t} \). The imaginary part,

\[
y = -\frac{21}{100} \cos 2t + \frac{3}{100} \sin 2t
\]

is a solution of \( y'' + 7y' + 6y = 3 \sin 2t \).

5.18. The homogeneous equation \( y'' + 3y' + 2y = 0 \) has characteristic equation \( \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0 \) with zeros \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \). This leads to the homogeneous solution

\[
y_h = C_1e^{-t} + C_2e^{-2t}.
\]

The particular solution \( y_p = Ae^{-4t} \) has derivatives \( y'_p = -4Ae^{-4t} \) and \( y''_p = 16Ae^{-4t} \), which when substituted into the equation \( y'' + 3y' + 2y = 3e^{-4t} \) provides

\[
16Ae^{-4t} + 3(-4Ae^{-4t}) + 2Ae^{-4t} = 3e^{-4t}
\]

\[
6A = 3
\]

\[
A = \frac{1}{2}
\]

Thus, a particular solution is \( y_p = (1/2)e^{-4t} \). This leads to the general solution

\[
y = C_1e^{-t} + C_2e^{-2t} + \frac{1}{2}e^{-4t}.
\]

The initial condition \( y(0) = 1 \) provides

\[
1 = C_1 + C_2 + \frac{1}{2}.
\]

Differentiating,

\[
y' = -C_1e^{-t} - 2C_2e^{-2t} - 2e^{-4t}.
\]

The initial condition \( y'(0) = 0 \) provides

\[
0 = -C_1 - 2C_2 - 2.
\]

This system has solution \( C_1 = 3 \) and \( C_2 = -5/2 \), leading to the solution

\[
y = 3e^{-t} - \frac{5}{2}e^{-2t} + \frac{1}{2}e^{-4t}.
\]
5.20. The homogeneous equation \( y'' + 2y' + 2y = 0 \) has characteristic equation \( \lambda^2 + 2\lambda + 2 = 0 \) with zeros \( \lambda_1 = -1 + i \) and \( \lambda_2 = -1 - i \). This leads to the homogeneous solution
\[
y_h = e^{-t}(C_1 \cos t + C_2 \sin t).
\]
The particular solution \( z = Ae^{2it} \) has derivatives
\[
z' = (2i)Ae^{2it}
z'' = (2i)^2 Ae^{2it}
\]
which, when inserted in the complex equation \( z'' + 2z' + 2z = 2e^{2it} \), gives
\[
((2i)^2 + 2(2i) + 2) Ae^{2it} = 2e^{2it}
\]
\[
A = \frac{2}{-2 + 4i} = \frac{1}{5} - \frac{2}{5}i.
\]
This gives the particular solution
\[
z = \left( \frac{1}{5} - \frac{2}{5}i \right) e^{2it}
z = \left( \frac{1}{5} - \frac{2}{5}i \right) (\cos 2t + i \sin 2t).
\]
The real part of this solution,
\[
y_p = \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t,
\]
is a particular solution of \( y'' + 2y' + 2y = 2 \cos 2t \). Thus, the general solution is
\[
y = e^{-t}(C_1 \cos t + C_2 \sin t) - \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t.
\]
The initial condition \( y(0) = -2 \) gives \(-2 = C_1 - 1/5\). Differentiating,
\[
y' = e^{-t}(-C_1 \sin t + C_2 \cos t) - e^{-t}(C_1 \cos t + C_2 \sin t) + \frac{2}{5} \sin 2t + \frac{4}{5} \cos 2t.
\]
The initial condition \( y'(0) = 0 \) provides
\[
0 = C_2 - C_1 + \frac{4}{5}.
\]
This system has solution \( C_1 = -9/5 \) and \( C_2 = -13/5 \). Therefore, the solution is
\[
y = e^{-t} \left( \frac{-9}{5} \cos t - \frac{13}{5} \sin t \right) - \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t.
\]

5.26. The homogeneous equation \( z'' + 4z = 0 \) has characteristic equation \( \lambda^2 + 4 = 0 \) and zeros \( \pm 2i \). Thus, the homogeneous solution is
\[
z_h = C_1 e^{2it} + C_2 e^{-2it}.
\]
The forcing term of \( z'' + 4z = 4e^{2it} \) is also a solution of the homogeneous equation, so multiply by a factor of \( t \) and try \( z_p = Ate^{2it} \). The particular solution has derivatives
\[
z'_p = Ae^{2it} (1 + 2it)
z''_p = 4Ae^{2it} (i - t).
\]
If these are substituted in \( z'' + 4z = 4e^{2it} \), then
\[
4Ae^{2it} (i - t) + 4Ate^{2it} = 4e^{2it}
\]
\[
(4(i - t) + 4t) A = 4
\]
\[
A = \frac{1}{t} = -i.
\]
Hence,

\[ z_p = -it e^{it} = it(\cos 2t + i \sin 2t) = t \sin 2t + it \cos 2t. \]

The real part of this solution is a particular solution of \( y'' + 4y = 4 \cos 2t \).

\[ y_p = t \sin 2t \]