# Chapter 3

# Formalization of Perceptual Requirements

In this chapter we show how it is possible to formalize perceptual requirements described in Chapter 2, use them together with additional technical restrictions to restrict the class of projections that we consider, and derive perceptual metrics that can help to evaluate projections from perceptual point of view.

## 3.1 Technical Requirements

To narrow down the area of the search for perceptually acceptable projections we are going to specify several additional design considerations. They don't have any perceptual basis and some of them are quite restrictive; however, they make the task of constructing projections manageable. Resulting projections can be applied to a wide class of images and the choice of correct tranformation can be made simple.

- We want a parametric family of viewing transformations so that an appropriate one can be chosen for each image.
- The number of parameters should be small, and they must have a clear intuitive meaning.
- Construction of the family of projections should not depend on the scene. Matching the scene to a particular projection in the family should be achieved by choosing an appropriate set of parameters.
- Pictures should be scalable: any part of a picture can be made a separate picture. Therefore, no area in a picture can be considered small enough to be ignored.

## 3.2 Structural Requirements

Structural distortions, as described in Section 2.1, are in general more apparent than nonstructural distortions. It is not difficult to find exceptions to this rule, but given the additional condition of scalability ( Section 3.1), almost any structural distortion would be highly objectionable.

This idea allows us to use structural and nonstructural requirements in different ways: we postulate that some structural requirements should be satisfied exactly, considerably reducing the variety of projections that we consider. Then nonstructural distortions can be minimized within the class of projections satisfying structural requirements.

Even given the scalability assumption, some structural requirements appear to be more important than others: if a requirement is associated with unusual and complicated object geometry, it is less important than those associated with simpler and more common objects.

In this section we choose several formalizations of certain structural requirements and apply them to images of *all* possible objects in *all* possible positions: we want our family of projections to be independent of the objects that we want to depict, and we want the resulting images to satisfy a given set of structural requirements exactly.

More specifically, we will show that three sets of structural requirements result in the same structure for the projections:

*the fibers of the projection should be subsets of the fibers of a perspective projection* (see Section 3.2.1 for the definition of a fiber.)

It doesn't mean that any perceptually acceptable projection necessarily has this structure. It rather means that if a projection doesn't have this structure, there are objects that will have unacceptable images if they are in certain positions in space.

For each set of structural requirements, we will prove the statement above by assuming the contrary (i.e. presence of curved or non-coplanar fibers) and construct an object that will have an image violating some structural requirement.

As the objects that we consider in Section 3.2.4 are quite simple and common (object with planar faces and straight edges) it seems reasonable to assume that most mappings with curved fibers are likely to produce considerable structural distortions for at least some scenes, and to concentrate in the rest of our study on the projections with the fiber structure of a perspective projection which guarantees absence of structural distortions.

### 3.2.1 Definitions and Preliminary Lemmas

We will use $x, y, ..$ for the points in the domain of a projection (a volume in 3D space), and $\xi, \psi ...$ for the points in the range (a point in the picture plane).

$\partial A$ denotes the boundary of the set $A$.

$\overline{A}$ denotes the closure of the set $A$.

**Definition 3.1** *By a* **line segment** *we mean any connected subset of a straight line.*

We will use the following notation for line segments:

– $[a, b]$ denotes the closed line segment between the points $a$ and $b$;

– $]a, b[$ denotes the open line segment between the points $a$ and $b$;

– $(a, b)$ denotes the straight line containing the points $a$ and $b$.

**Definition 3.2** *We will call a mapping $P : \mathbf{R}^n \rightarrow R^m$ smooth, if all the components of this mapping are at least twice continuously differentiable.*

**Definition 3.3** *The set of all points of the domain of a mapping that map to a fixed point $\xi$ is called the* **fiber** *of the mapping at the point $\xi$.*

If a fiber is a line segment and is not a point, we will call the straight line containing it **the fiber line**. For each point $x$ in the domain of a mapping there is fiber going through this point. We will denote it $F(x)$.

**Definition 3.4** *A* **projection** *is a smooth mapping $P : V \subset \mathbf{R}^3 \rightarrow \mathbf{R}^2$, where $V$ is an open path-connected domain in $\mathbf{R}^3$, with the rank of the Jacobian of $P$ equal to 2 in all points of $V$.*

**Definition 3.5** *We will call a continuous (smooth) curve $\gamma_\epsilon : I \rightarrow \mathbf{R}^3$, where $I$ is a line segment, an* **$\epsilon$-perturbation** *of a continuous (smooth) curve $\gamma : I \rightarrow \mathbf{R}^3$ if there is a homeomorphism (diffeomorphism) $g : \gamma_\epsilon(I) \rightarrow \gamma(I)$ such that the distance $|g(x) - x|$ is less than $\epsilon$ for any $x \in \gamma(I)$.*

**Definition 3.6** *A* **homotopy** *of a continuous curve $\gamma : I \rightarrow A$ to another curve $\gamma' : I \rightarrow A$ in $A \subset \mathbf{R}^n$ is a continuous mapping $h : [0, 1] \times I \rightarrow A$, such that $h(0, t) = \gamma(t)$ and $h(1, t) = \gamma'(t)$ for all $t \in I$.*

**Definition 3.7** *Two smooth planar curves intersect* **transversally** *at a point $x$, if their tangents at this point are not parallel. A smooth curve and a smooth surface intersect* **transversally** *if the tangent vector of the curve is not in the tangent plane of the surface.*

A **crossing** is an intersection of two curves that cannot be eliminated by an arbitrarily small perturbation of the curves. More precisely,

**Definition 3.8** *Two planar curves in an open set $A \subset \mathbf{R}^2$ have a* **crossing** *inside $A$ if they have an intersection point in $A$ and there is $\epsilon$ such that for any $\epsilon'$-perturbation of any of the intersecting curves, where $\epsilon' < \epsilon$, the perturbed curves still intersect inside $A$.*

If two curves intersect transversally in $A$ they have a crossing in $A$. Some non-transversal intersections result in crossing; some don't (Figure 3.1.)

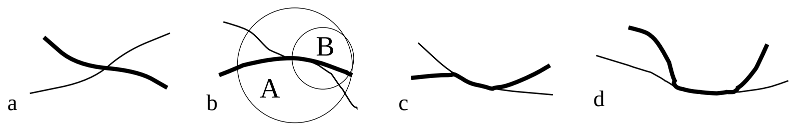


Figure 3.1: Examples of intersections. a. Transversal; curves $\gamma$ and $\gamma'$ have a crossing in any open set containing the intersection. b. Nontransversal; curves $\gamma$ and $\gamma'$ still have a crossing in any open set containing the intersection. c. nontransversal; $\gamma$ and $\gamma'$ have a crossing in $A$ and don't have a crossing in $B$. d. nontransversal, $\gamma$ and $\gamma'$ don't have a crossing in any set. □

Two closed curves are homotopic in a set if one can be continuously deformed to the other without leaving the set.

The following definition is a mathematical expression for "having a different number of holes." We will use without proof the fact that a smooth closed curve without self-intersections separates the plane into two path-disconnected sets - one bounded ("inside") and the other unbounded ("outside"). The curve can be retracted to a point by a homotopy in the "inside" area and cannot be retracted to a point in the "outside" area.

**Definition 3.9** *We will define a* **hole** *in the image $P(O)$ of an object $O$ to be a path-connected set of points $H$ that don't belong to $P(O)$, if in $P(O)$ there is a closed smooth curve without self-intersections such that $H$ lies inside the curve. Two holes are different if there is a smooth closed curve without self-intersections such that one hole lies inside the curve and the other outside.*

**Definition 3.10** *The* **convex hull** *of a set is the union of all line segments connecting two points from this set.* $\mathrm{CH}(A)$ *denotes the convex hull of the set $A$.*

Now we prove several lemmas that we will use later.

**Lemma 3.1** *If the points of a continuous curve $\gamma : [0,1] \to \mathbf{R}^3$ are divided into two subsets $S$ and $S'$ so that $S \cap S' = \emptyset$, $S \cup S' = \gamma([0,1])$ then there is a common limit point of $S$ and $S'$.*

**Proof.**

Suppose for each point of $S$ and $S'$ there is a neighborhood which doesn't contain any points of the other set. Let $x \in S, x' \in S'$. Then there is a continuous curve from $x$ to $x'$. Consider the inverse images $I = \gamma^{-1}(S)$ and $I' = \gamma^{-1}(S')$. It follows from our assumption that for each point in $I$ there is a neighborhood not containing points in $I'$. Let the union of all these neighborhoods be $N$ and the union of neighborhoods constructed in the same way for $S'$ be $N'$. Then $N$ and $N'$ are open sets which cover the whole interval. Their intersection is an open set that doesn't contain any points of $I$ and $I'$; therefore it is empty ($I \cup I' = [0,1]$.) Thus we have a decomposition of the interval into two nonempty open sets which is impossible, as the interval is connected.

■

**Lemma 3.2** *If $P : V \subset \mathbf{R}^3 \to \mathbf{R}^2$ is a continuously differentiable mapping with the rank of the Jacobian everywhere equal to 2, then*

*- any fiber $P^{-1}(\xi)$ of the mapping is a 1-dimensional submanifold of $\mathbf{R}^3$;*

*- the unit tangent vector to this manifold at any point $x$ is a continuously differentiable function of $x$ in a neighborhood of $x$ for some choice of orientations;*

*- the curvature of the fiber is a continuous function of $x$.*

The first part of the statement immediately follows from Theorem 1.38 in [Warner, 1983]. Let's prove the second and the third part.

Consider a local one-to-one parametrization of $P^{-1}(\xi)$ in a neighborhood of a point $x_0 = (x_0^1, x_0^2, x_0^3)$, $x = x(t)$. The tangent vector at a point $x$ is $\dot{x}(t) \neq 0$. $P(x(t)) = 0$. Differentiating this expression with respect to $t$, we get

$$\frac{dP(x)}{dx}\dot{x}(t) = 0$$

As the Jacobian $\frac{dP(x)}{dx}$ has rank 2, solution of this system exists and is unique up to a constant. Therefore the unit tangent vector $\tau$ is defined uniquely by the system and the formula

$$\tau = \frac{\dot{x}(t)}{|\dot{x}(t)|}$$

Components of $\tau$ are continous functions of the partial derivatives of the components of $\frac{dP(x)}{dx}$. As $P$ is assumed to be continuously differentiable, the derivatives are continuous functions of $x$ Therefore, $\dot{x}(t)$ is a continuous function of $x$.

As the curvature is simply the length of $\ddot{\tau}(t)$, it can be proven to be continuous by differentiating the expression for $\tau$. It is important to note that $\dot{x}(t)$ doesn't depend explicitly on $t$.

■

**Lemma 3.3** *If all fibers of a projection are line segments and any two fibers are coplanar the following is true: if any two fiber lines intersect but do not coincide, all fiber lines intersect at the same point. Otherwise, all fiber lines are parallel.*

**Proof.** Let $f_1$ and $f_2$ be two intersecting but not coinciding fiber lines. Let $x \in V$ be a point outside the plane $p$ defined by $f_1$ and $f_2$, and $f_3$ be the fiber line of $x$. As $f_3$ should be coplanar to $f_1$ it can be either parallel to it or intersect it. But if it is parallel to $f_1$, it is parallel to the plane $p$, and therefore doesn't intersect $f_2$. But it isn't parallel to $f_2$ either; therefore, $f_3$ and $f_2$ are not coplanar. Thus, $f_3$ should intersect $f_1$ and $f_2$. But it can intersect the plane $p$ only in one point, so it should be the common point of $f_1$ and $f_2$.

Consider a point $x'$ in the plane $p$ and the fiber line $f_4$ corresponding to it. The same reasoning applies to $f_4$ and the pair $f_1$, $f_3$.

Suppose $f_1$ and $f_2$ do not intersect. Suppose some fiber $f_3$ intersects $f_1$. If it is not in $p$, then it cannot be parallel to $f_2$ and it doesn't intersect it, which is impossible. If it is in $p$, then a fiber line $f_4$ which is not in the plane should intersect it in two points: $f_1 \cap f_3$ and $f_2 \cap f_3$, which is impossible. ■

### 3.2.2 Holes in Images

Given our definition of holes, description of example a on Figure 2.4 obtains exact meaning:

> **Condition H.** An image of an object should not have a number of holes different from the number of holes for any perspective projection of the object.

**Lemma 3.4** *Given Condition H, all fibers of the projection should be line segments.*

**Idea of the Proof.** Suppose there is a fiber which is not a linear segment. We construct a tube around this fiber in such a way that no straight line can get through the tube. Therefore, no perspective projection of the tube can have holes in it. The fiber goes through the tube without intersecting the walls, therefore, there is at least one hole in the image of the tube under $P$, which contradicts the Condition H.

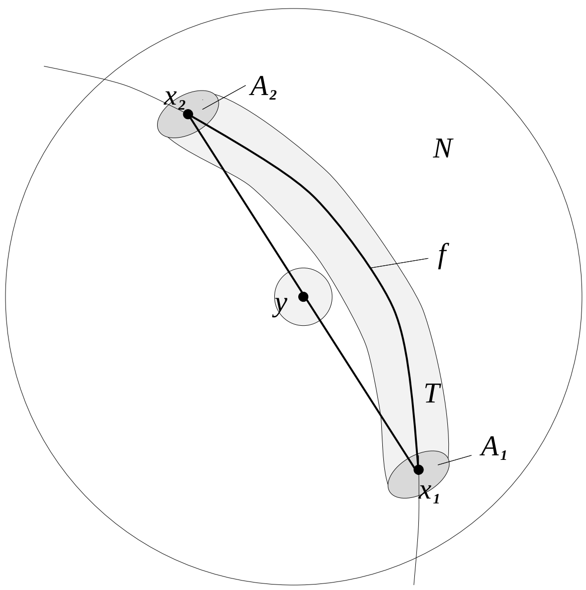


Figure 3.2: Construction of Lemma 3.4 □

**Proof.**

Suppose a fiber $f$ is not a line segment. Choose a point $x$ on the fiber where the curvature is not zero. There is a neighborhood of the point $x$ where the curvature is not zero, because the curvature is continuous (Lemma 3.2.)

As the fiber $f$ is a one-dimensional submanifold of $\mathbf{R}^3$ with induced topology, we can find an open neigborhhood $N_f$ of $x$ in $f$ which is diffeomorphic to an open interval and $N_f = N \cap f$, where N is an open set in $\mathbf{R}^3$. We can choose $N_f$ and $N$ so that $N$ is a ball.

Consider the endpoints $x_1$ and $x_2$ of $N_f$. If $N_f$ is chosen in the interior of $f$, they should be outside $N$. As they are limit points of $N_f$, they should be on $\partial N$. These are the only intersections of $f$ with $\partial N$.

Let $f_c$ be the closure of $N_f$. Consider a cover of $f_c$ with open balls centered at the points of $f_c$. Choose the balls so that for the points of $N_f$ their closures are inside $N$. As $f_c$ is bounded and closed, it is compact. Therefore, there is a finite subcover $\mathrm{Cover}_{f_c}$, and there is $\epsilon$ such that any ball of the radius less than $\epsilon$ centered at a point of $f_c$ is inside $\mathrm{Cover}_{f_c}$. Let $T = \mathrm{Cover}_{f_c} \cap N$.

$T$ has the following properties: $T \cap f = N_f$, $f \cap \partial T = \{x_1, x_2\}$. If a point $y$ is inside $T$, there is a point $y_f$ of $f$ such that $|y - y_f| < \epsilon$. As $N$ is a ball, the convex hull CH($T$) will be contained in $N$ and CH($\overline{T}$) will be contained in $\overline{N}$. Therefore, $\mathrm{CH}(T) \cap f = N_f$.

Consider the line segment $[x_1, x_2]$. As the curvature of $f$ in $N$ is not zero, ther is a point $y$ on this line segment which is not on $f_c$. there are neigborhoods $M_f$ of $f_c$ and $M_y$ of $y$ which don't intersect. Choose $\epsilon$ small enough for $T$ to be inside $M_f$ and for the ball of radius $\epsilon$ to be inside $M_y$.

Define the object $O'$ to be CH($\overline{T}$). Consider $\partial T \cap \partial N$. By construction, the only part of $\partial T$ that can intersect $\partial N$ should belong to the boundary of the balls $B(x_1)$ and $B(x_2)$ centered at $x_1$ and $x_2$. Let $A_1 = B(x_1) \cap \partial N$, $A_2 = B(x_2) \cap \partial N$. For sufficiently small $\epsilon$, $A_1$ and $A_2$ are disjoint, i.e. $\overline{T} \cap \partial N$ is a disjoint union of $A_1$ and $A_2$. As $A_1$ and $A_2$ are intersections of a ball and a sphere, $\partial A_1$ and $\partial A_2$ in $\partial N$ are circles. Define the object $O$ as $(O' \setminus T \setminus (A_1 \cap A_2)) \cup \partial A_1 \cup \partial A_2$. ( We throw away $T$, and the interior of the intersection of $\overline{T}$ with $\partial N$.)

Consider a perspective projection $Q$ with center outside $O$. Let $l$ be a straight line going through the center of projection (a fiber line of the perspective projection) and let $\xi$ be the corresponding point in the picture plane. We want to show that there are no holes in the image of $O$. As $O'$ is convex, its perspective projection image is convex too, and it is easy to show that it doesn't have holes. As $Q(O) \subset Q(O')$, no point that doesn't belong to $Q(O')$ can be a point of a hole in $Q(O)$.

We have to consider only the points of $Q(O') \setminus Q(O)$. If $\xi \in Q(O') \setminus Q(O)$, the line $l$ intersects $O'$ but doesn't intersect $O$, i.e. intersects $O'$ inside $T$.

No point outside $T$ can be connected by a continuous curve to a point in $T$ without crossing $\partial T$. All the points of $\partial T$, except those that are in $\partial N$, are in $O$. If the straight line $l$ intersects $O'$, but doesn't intersect $T$, it should go through the points of $A_1$ or $A_2$.

Suppose $l$ intersects only one of these sets. Consider a plane through $l$ which intersects $\partial N$ only inside $A_1$. Then all the points of $O$ are on one side of this plane (the only points of $O'$ that are on the other side belong to $T$.) Therefore, for any curve in $O$ there is a homotopy in $\mathbf{R}^3$ that contracts it to a point, without intersecting $l$. The image of $O$ has the same property with respect to the image $\xi$ of $l$, which is a point. Therefore, there is no closed curve in $Q(O)$ that has $\xi$ inside, and $\xi$ cannot belong to a hole in $Q(O)$.

Suppose $l$ intersects both sets. Any line going through the balls of radius $\epsilon$ centered at $x_1$ and $x_2$ goes through a ball of radius $\epsilon$ centered at $y$. But this ball doesn't intersect $T$, therefore, there is a point on $l$ which is outside $T$ and inside $\partial N$. Then the line should intersect $\partial T \setminus \partial N$, which is part of $O$. Thus, $\xi$ is in the image of $O$.

We conclude that $Q(O)$ has no holes in it. $P(O)$, however, has a hole. Fiber $f$ doesn't intersect $O$ by construction. It intersects the set $A_1$ at the point $x_1$. Consider the image of $A_1 \cup \partial A_1$. $\partial A_1$ is a circle, therefore, its image can be considered as a smooth closed curve. Again, if $\epsilon$ is sufficiently small, we can assume that the directions of the fibers going through the points of $A_1$ are within a small solid angle and are transversal to the sphere $\partial N$. The image of $A_1 \cup \partial A_1$ is one-to-one, and the curve $S^1 \to P(\partial(A_1))$ doesn't have self-intersections.

Clearly, $x_1 \in P(A_1)$ is in the interior of this curve, and therefore, there is a hole in the image of $O$.

Threfore, $P(O)$ and $Q(O)$ have different number of holes for any choice of $Q$ and Condition H is violated.

■

**Lemma 3.5** *Given Condition H, all fibers should be coplanar.*

**Idea of the Proof.** The idea is similar to the proof of the previous lemma. If there are two non-coplanar fibers, we can construct an object with tubes through the object along the fibers in such a way that one cannot see through both tubes at the same time. There will be at least two holes in the image of the object under $P$, and no more than one under $P$.

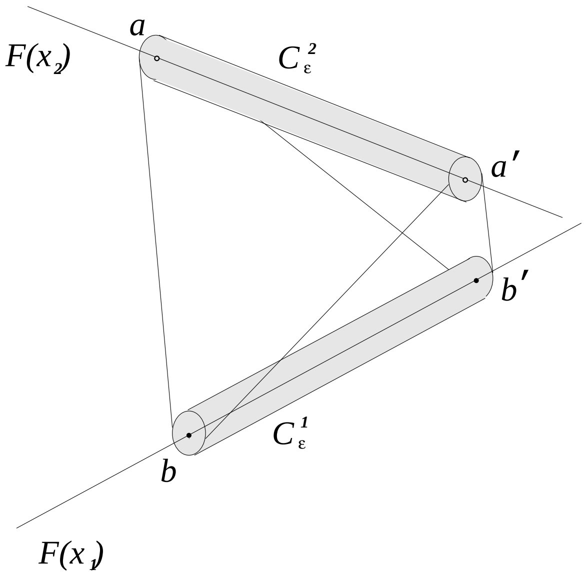


Figure 3.3: Construction from Lemma 3.5 □

**Proof.**

Suppose we have two non-coplanar fibers, $f_1$ and $f_2$.

As we have proven, all the fibers are line segments. Choose a curve $\gamma$ connecting two points on the fibers $f_1$ and $f_2$. We can separate all the points on the path into two sets: $S$, the set of all points for which the fiber $F(x)$ is coplanar with $f_1$, and $S'$, the rest of the path. Using Lemma 3.1 we can find a point $x$ such that in any neighborhood of this point there are points from $S$ and $S'$. Let $B(x)$ be a ball contained in $V$. Choosing $x_1$ to be a point from $S$ and $x_2$ to be a point from $S'$, we obtain fibers $F(x_1)$ and $F(x_2)$ that are not coplanar and both intersect $B(x)$. We can pick points $a, a' \in F(x_1)$ and $b, b' \in F(x_2)$ inside $B(x)$.

Consider two cylinders $C_\epsilon^1$ and $C_\epsilon^2$ of radius $\epsilon$ and with axes $[a, a']$ and $[b, b']$. Let $l_1$ and $l_2$ be line segments connecting points on the opposite faces of $C_\epsilon^1$ and $C_\epsilon^2$ respectively. We can choose $\epsilon$ in such a way that for any choice of $l_1$, $l_2$ they are not coplanar and $C_\epsilon^1$ and $C_\epsilon^1$ are inside $B(x)$. Appropriate choice of $a$, $a'$, $b$ and $b'$ can also ensure that the faces of the cylinders are inside the boundary of the convex hull $CH(C_\epsilon^1 \cup C_\epsilon^2)$.

Consider an object $O$ defined as $CH(C^1_\epsilon \cup C^2_\epsilon) \setminus (Int(C^1_\epsilon) \cup Int(C^2_\epsilon))$. Any perspective projection of this object has at most one hole: a straight line which intersects the convex hull, but doesn't intersect $O$, should enter at one of the cylinders' faces and exit at the opposite face of the same cylinder. If there are two disconnected regions in the image in which fibers of the perspective projection behave in this way, we have two coplanar straight lines (fibers of a perspective projection are coplanar) that intersect $CH(C^1_\epsilon \cup C^2_\epsilon)$ only inside $C^1_\epsilon$ and $C^2_\epsilon$ respectively. But by construction this is impossible. It can be shown that each cylinder can produce no more than one hole in a perspective projection.

On the other hand, as $[a, a']$ and $[b, b']$ belong to the fibers, and the boundaries of the faces of cylinders map to curves enclosing the points $P(a)$ and $P(b)$, the image $P(O)$ has at least two holes. ■

By Lemma 3.3, all fiber lines are either parallel or intersect at one point. Therefore, fibers of $P$ are subsets of fibers of a perspective projection.

The main problem with the argument above is that the objects that we construct are rather unusual and the resulting structural distortion is not very significant: it is not obvious when we are supposed to be able to see through a curved pipe and when we are not. In the following subsections we will to consider simpler objects.

### 3.2.3 Loops, Folds and Twists

In this section we consider the following conditions, which formalize (Figure 2.4 d,e), but are more restricitve. We describe this set of conditions mostly for historical reasons –this is the set described in [Zorin and Barr, 1995]. The conditions used in the next section formalize essentially the same requirements in a better way.

The main advantage of this formalization is that we don't have to deal with smoothness properties and the proofs become simpler.

> **Condition L.** The mapping of any line segment $l$ to its image $P(l)$ is one-to-one everywhere or it is a point.

This condition prevents "loops" in the images of lines. It is more restrictive: not only it doesn't allow "loops" but also "folds", that is situations when the image is not one-to-one, but is still simply connected.

**Condition T.** The mapping of a subset of a plane $m$ to the image $P(m)$ is one-to-one everywhere or nowhere.

This condition prevents "twists" in the images of planes.

**Lemma 3.6** *Given condition L, all fibers of the mapping $P$ are line segments.*

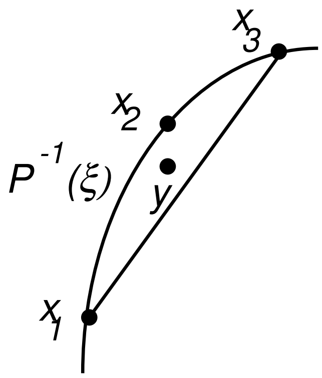


Figure 3.4: Construction from Lemma 3.2.3 □

**Proof.**

Consider a fiber $P^{-1}(\xi)$. Suppose it is not a line segment. Let $x_1$ and $x_2$ be two points of the fiber. Then $P^{-1}(\xi) \not\subset (x_1, x_2)$.

Let $S = P^{-1}(\xi) \cap (x_1, x_2)$, $S' = P^{-1}(\xi) \setminus (x_1, x_2)$. It is possible to show that there is a point $x \in P^{-1}(\xi)$ such that in any neighborhood of this point there are points of $S$ and $S'$. Let $B(x)$ be a ball contained in $V$ (the domain of the mapping). We can choose $x_1$, $x_2$ and $x_3 \in P^{-1}(\xi)$ to be three non-collinear points in $B(x)$ .

Consider the image of the triangle $\triangle x_1\ x_2\ x_3$. As the whole triangle cannot map into $\xi$ (this would imply that $\dim P^{-1}(\xi) > 1$, contradicting Condition L, there is a point $y$ in the triangle such that $P(y) = \xi' \neq \xi$. Consider the segment $[x_1, y'] \ni y$, where $y' \in [x_2, x_3]$. If $P(y') = \xi$ then the image of $[x_1, y']$ is not one-to-one but contains at least two points. If $P(y') \neq \xi$, then the same is true for $[x_2, x_3]$. Thus the assumption that a fiber can contain non-collinear points contradicts Condition L.

■

Note that this lemma doesn't rely on the Condition T.

**Lemma 3.7** *Any two fibers are coplanar.*

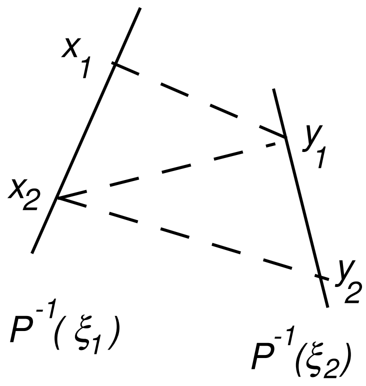


Figure 3.5: Construction from Lemma 3.7 □

**Proof.**

Suppose two fibers are not coplanar. Choose two points on these fibers. Consider a continuous path connecting these two points that is contained in $V$. Such a path exists because $V$ is assumed to be path-connected. As in the previous lemma, we can separate all the points on the path into two sets: $S$, the set of all points $x$ for which the fiber $P^{-1}(P(x))$ is coplanar with $P^{-1}(P(x_1))$, and $S'$, the rest of the path. In the same manner we can find a point $x$ such that in any neighborhood of this point there are points from $S$ and $S'$. Again, let $B(x)$ be a ball contained in $V$. Let $P^{-1}(\xi_1)$ and $P^{-1}(\xi_2)$ be two non-coplanar fibers corresponding to points in $B(x)$. Consider two points $x_1$ and $x_2$ on $P^{-1}(\xi_1)$ and $y_1$ and $y_2$ on $P^{-1}(\xi_2)$. The image of $[x_1, x_2]$ is not one-to-one; it follows from Condition T, that the image of $\triangle x_1\ x_2\ y_1$ is not one-to-one anywhere. Let $\xi'$ be a point in the image. As the intersection of the fiber $P^{-1}(\xi')$ with the plane of the triangle consists of more than one point, and the fiber is a line segment, the whole fiber belongs to the plane of the triangle. But the same thing will be true for $\triangle y_1\ y_2\ x_1$. Choose $\xi' \in P([x_1, y_1])$. As two triangles share the edge $[x_1 y_1]$, $\xi' \in P(\triangle y_1\ y_2\ x_1)$ and $\xi' \in P(\triangle x_1\ x_2\ y_1)$, and the fiber $P^{-1}(\xi')$ should be in the planes of both triangles, i.e. in $(x_1, y_1)$. As it follows from continuity, should contain $[x_1, y_1]$. This is impossible, because $x_1$ and $y_1$ map to different points. Therefore, there can be no two fibers that are not coplanar.

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As before, from the two lemmas that we have proven, it follows that the fibers of$P$ are subsets of fibers of a persepective projection.

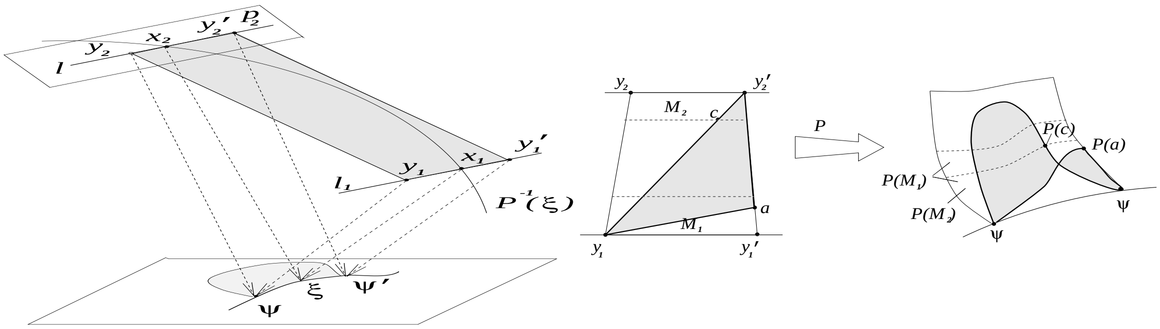


Figure 3.6: Constructions from Lemma 3.8: the image of the triangle $y_1$ $a$ $y_2'$ is twisted □

### 3.2.4 Loops and Twists

The following statement is the precise expression of the idea that the image of a line-segment should be a curve with no loops (although not necessarily one-to-one.)

**Condition L′.** The image of a line segment should be a simply connnected curve.

"Twists" in the image of a plane can be described more precisely in the following way: suppose an object has a flat face with several straight edges intersecting only at the endpoints. In this case no perspective projection produces images of edges crossing each other at some point in the middle unless they coincide. This leads us to the following formalization:

**Condition T′.** The image of a pair of coplanar line segments should have no crossing points that are not images of an intersection point of the segments.

Again, we prove the same two types of lemmas as in previous sections.

**Lemma 3.8** *Given Conditions $L'$ and $T'$, every fiber is a line segment.*

**Proof.**

Suppose a fiber $P^{-1}(\xi)$ has non-zero curvature in a neighborhood of a point.

Choose two points $x_1$, $x_2$ on the fiber $P^{-1}(\xi)$ inside this neighborhhood. Consider two parallel line segments $l_1$ and $l_2$ going through the points $x_1$ and $x_2$ respectively, $l_1, l_2 \subset V$. Choose these intervals so that $x_1$ and $x_2$ are their interior points and the plane through $l_1$ and $l_2$ is transversal to all the fibers going through $l_1$ and $l_2$.

The images of the line segments $l_1$ and $l_2$ intersect at the point $\xi$. Suppose $\xi$ is an intersection point such that there is no (possibly one-sided) neighborhood of $\xi$ where the curves $P(l_1)$ and $P(l_2)$ coincide.

There should not be a crossing (Condition T$'$), so the intersection at $\xi$ shouldn't be transversal. Consider a plane $p_2$ through the line segment $l_2$, which is crossed by $P^{-1}(\xi)$ transversally. The fibers of the mapping $P$ going through the line segment $l_1$ map it into a smooth curve in the plane $p_2$ which is a one-dimensional submanifold (Condition L$'$.) Therefore, the tangent vector to the image of $l_1$ is unique at all points. As the plane $p_2$ is transversal to the fiber $P^{-1}(\xi)$, we can choose a one-dimensional neighborhood of $x_1$ in $l_1$ in such a way that for each fiber going through a point in this neighborhood there is only one intersection with the plane $p_2$ (this is possible due to continuity of the tangent vector, Lemma 3.2.) Restriction $P|_{p_2}$ of the mapping $P : \mathbf{R}^3 \to \mathbf{R}^2$ to the plane $p_2$ is differentiable and the point $x_2$ is nonsingular, because of the transversality assumption. Therefore, in a neighborhood of $x_2$ it can be inverted and the inverse mapping will have the same smoothness (inverse function theorem.) Consider $T = P|_{p_2}^{-1}$. It maps the neighborhhood $N$ of $\xi$ to a neighboorhood $N_2$ of $x_2$, the curve $P(l_1)$ to a smooth curve in the plane $p_2$. As $\xi$ is an intersection point such that there is no (possibly one-sided) neighborhood of $\xi$ where the curves $P(l_1)$ and $P(l_2)$ coincide, the same is true for $x_2$, $T(l_1)$ and $l_2$ in $p_2$. Then there is a point $x_2$ in $N_2$ on $T(P(l_1))$ where the tangent vector is not parallel to $l_2$. Consider $l_2'$ which lies in the plane $p_2$ and goes through the point $x_2'$. The curve $T(P(l_1))$ intersects $l_2'$ at that point and is transversal to it. Therefore, the same is true for the images $P(l_1)$ and $P(l_2)$, contradicting Condition T$'$. We conclude that any intersection point of $P(l_1)$ and $P(l_2)$ must have a neighborhood, possibly one-sided, where the curves coincide.

Pick two points $\psi$ and $\psi'$ in this neighborhood. Let $y_1 = P^{-1}(\psi) \cup l_1$, $y_1' = P^{-1}(\psi') \cup l_1$, $y_2 = P^{-1}(\psi) \cup l_2$, $y_2' = P^{-1}(\psi') \cup l_2$.

Consider the planar quadrangle $y_1\ y_1'\ y_2'\ y_2$. The images of the edges $[y_1, y_1']$ and $[y_2, y_2']$ coincide by construction.

For every point of $l_1$ there is a planar neighborhood in the plane $p_q$ of the quadrangle $y_1\ y_1'\ y_2'\ y_2$ where the mapping $P|_q$ is one-to-one because at the points of $l_1$ fibers are transversal to the plane by construction. Due to the compactness of $l_1$, there is an $\epsilon$ such that there are no such points that are closer than $\epsilon/2$ to $[y_1, y_1'] \subset l_1$. Pick a point $a$ on $[y_1', y_2']$ which is closer to $l_1$ than $\epsilon$. Denote points on $[y_1, y_2]$ and $[y_1', y_2']$ which are at the distance $\epsilon/2$ from $y_1$ and $y_1'$ as $b_1$ and $b_1'$. Denote the quadrangle $y_1, y_1', b_1, b_1'$ as $M_1$. In the same way we can construct the quadrangle $M_2$ near $l_2$. We can choose $M_1$, $M_2$ and $a$ in such a way that $P(M_2) \subset P(M_1)$ and $P([y_1, a]) \in P(M_2)$.

Consider the line segments $[y_1, y_2']$ and $[y_1, a]$.

The image of $[y_1, y_2']$ is a curve connecting $P(y_1)$ and $P(y_2') = P(y_2)$. $P([y_1, y_2'])$ crosses $P([b_2, b_2'])$ at a unique internal point $P(c)$ because the mapping of $M_2$ to the picture plane is one-to-one and $[y_1, y_2']$ intersects $[b_2, b_2']$ at one interior point $c$. $P(]c, y_2'[)$ lies in the interior of $P(M_2)$ and connects two points on the boundary. $P(]y_1, a[)$ lies in the interior of $P(M_2)$ and connects two points on the boundary.

We can easily show that $M_1 \setminus [y_1, a]$ is separated by $[y_1, a]$ into two sets $S$ and $S'$ with the following property: there is a path connecting two points of $M_1$ and lying inside $M_1$ that doesn't intersect $[y_1, a]$ if and only if the points belong to the same set. Define these sets to be the sets of points on the different sides of the line $(y_1, a)$ in the plane of the quadrangle. Consider two points $s$ and $s'$ from different sets and a connecting path $\gamma(t)$. Choose a unit normal vector $\mathbf{n}$ to the line $(y_1, a)$. Let $\mathbf{d}$ be the vector from $y_1$ to $\gamma(t)$. Consider the scalar product $(\mathbf{d}, \mathbf{n})$. If it is negative for $s$, it should be positive for $s'$. Therefore, there is a point on the curve where it is zero (continuity) and so the curve intersects $[y_1, a]$.

This property transfers to $P(M_1)$ and to $P(M_2)$, due to the inclusion $P(M_2) \subset P(M_1)$. The point $P(c)$ belongs to $P([b_2, b_2'])$. As $P([b_2, b_2'])$ is part of the boundary, and the only boundary points of $P([y_1, a])$ are $P(y_1)$ and $P(a)$, they don't intersect, as $P|_{M_2}$ is one-to-one. Therefore, $P([b_2, b_2'])$ lies entirely in one of the sets $P(S)$ or $P(S')$. It also lies in the same set as $P(b)$ because $P(b)$ and $P(b_2)$ can be connected by a path that doesn't cross $P([y_1, a])$. The point $P(y_2')$ lies in the other set, as $y_1$ and $b_1$ are in different sets and $P(y_1') = P(y_2')$. We conclude that $P([y_1, a])$ should intersect $P([c, y_2'])$ at an interior point. This intersection is a crossing: take interior points $s'$ and $s$ of $P(S')$ and $P(S)$ lying on the curve $P(y_1, a])$. There are neighborhoods of these points that are contained in $P(S')$ and $P(S)$ respectively. Thus, if the perturbation doesn't take $s'$ and $s$ of the curve out of these neighborhoods and doesn't take the part of the curve between $s$ and $s'$ out of $P(M_2)$, then the intersection is not eliminated. This is the case for sufficiently small perturbations, because the part of the curve between $s$ and $s'$ lies entirely inside $P(M_2)$. The same is true for $P([c, y_2'])$. Therefore, $P(y_1, a])$ and $P([c, y_2'])$ violate Condition T$'$, and we get a contradiction. Therefore, all fibers should have zero curvature, i.e.all fibers should be line segments.

■

**Lemma 3.9** *Given Conditions $L'$ and $T'$, all fibers are coplanar.*

**Proof.**

Assume the that the opposite is true. In the same way as in Lemma 3.2.3 we construct two continuous parts of non-coplanar fibers $f_1$ and $f_2$ inside a ball which lies completely in $V$. Choose two points $x_1$ and $x_2$, on fibers $f_1$ and $f_2$ respectively.

Choose a point $x$ on $[x_1, x_2]$ so that if we draw a plane $p$ through the fiber going through $x$ and $[x_1, x_2]$ the fibers $f_1$ and $f_2$ won't lie in the plane.

The tangent vector to the fiber is a differentiable function of the point in $\mathbf{R}^3$. Therefore, the angle between a fixed direction and the fiber $F(x(t))$, where $x(t)$ is a parametrization of $[x_1, x_2]$, is a differentiable function. We can choose the direction so that it is not a constant. Then there is a line segment inside $[x_1, x_2]$ where the derivative of the angle with respect to $t$ has a constant sign. Then on this line segment all fibers have different angles with the fixed direction.

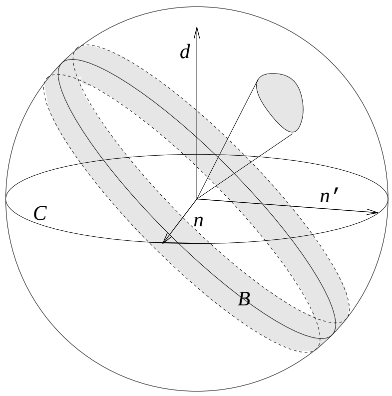


Figure 3.7: Unit sphere construction from Lemma 3.9 □

In addition, we can choose $x_1$ and $x_2$ close enough to ensure that no fibers along $[x_1, x_2]$ lie in the plane $p'$ which is perpendicular to $p$ and goes through $[x_1, x_2]$. The following construction on the unit sphere where points correspond to the directions of the fibers helps to explain why we can make such choice of $x_1$, $x_2$, $p$ and $p'$.

The directions of fibers fill a closed area $A$ on the unit sphere. This area doesn't include the point corresponding to the direction of $[x_1, x_2]$: $x_1$ and $x_2$ belong to different fibers. Let $\mathbf{d}$ be a unit vector corresponding to the direction of $(x_1, x_2)$, with an arbitrary choice of orientation.

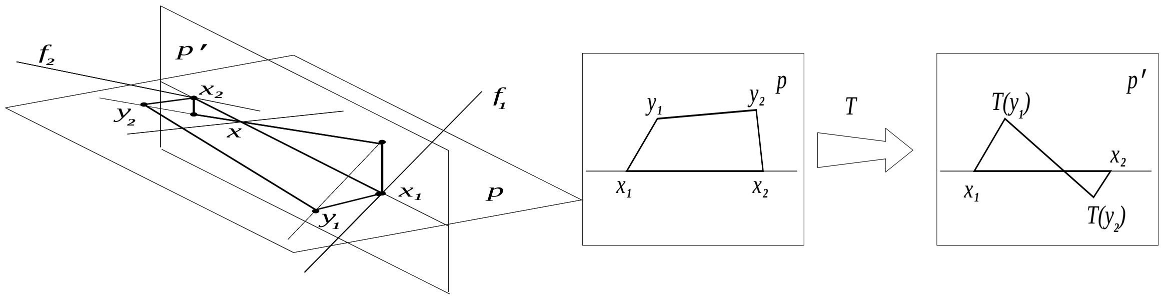


Figure 3.8: Construction from Lemma 3.9: the image of the quadrangle $x_1\ x_2\ y_2\ y_1$ is twisted. □

Possible directions of the normals $\mathbf{n}$ and $\mathbf{n}'$ of the planes $p$ and $p'$ form a circle $C$ in the plane perpendicular to $\mathbf{d}$ with arbitrary choice of orientation.

Directions perpendicular to $\mathbf{d}$ form a "band" $B$ –a union of the large circles of the sphere. This band doesn't include the circle of possible directions of $p, p'$ because $A$ doesn't include $\mathbf{d}$. As a result, the intersection of $B$ and $C$ can be made arbitrarily small. This intersection consists of two connected parts coresponding to sets of opposite directions. By making them sufficiently small, we can guarantee that the directions perpendicular to those in $C \cap B$ are not in $B$. Then we can choose $\mathbf{n}$ in $C \cap B$ and $\mathbf{n}'$ in $C$ perpendicular to $\mathbf{n}$. This will satisfy the conditions stated above.

Choose a half-space with respect to the plane $p$ and project the parts of the fiber contained in this half-space into $p$. We can always choose $p$ in such a way that these projections are on different sides of $(x_1, x_2)$: the point corresponding to $n$ in the circle $C$ should be between the points of intersection of large circles corresponding to the perpendiculars to the directions of $f_1$ and $f_2$ with $C$.

The plane $p'$ is transversal to the fibers in some neighborhood $N'$ of $[x_1, x_2]$. Therefore, the mapping of this neighborhood to the picture plane will be one-to-one and smooth. On the plane $p$ we can choose a neighborhood $N$ of $[x_1, x_2]$ which maps into the image of the neighborhood $N'$.

Choose the point $y_1 \in N$ on the projection of the fiber $f_1$ into the plane $p$ and $y_2 \in N$ on the projection of the fiber $f_2$, so that $y_1$ and $y_2$ are on one side of $(x_1, x_2)$. Choose the pairs of points in such a way that the intervals $[y_1, x_1]$ and $[y_2, x_2]$ don't intersect.

Let's show that there is a neighborhood of $[x_1, x_2]$ such that all fibers that lie in the plane $p$ and intersect this neighborhood, also intersect $[x_1, x_2]$. Suppose for any $\epsilon$ there is a fiber $f$ that lies in $p$, intersects $(x_1, x_2)$ at a point outside $[x_1, x_2]$ and goes through a point which is closer than $\epsilon$ to $[x_1, x_2]$. By construction, there is a fiber $f'$ intersecting but not containing $[x_1, x_2]$ that lies in the plane $p$. By choosing $\epsilon$ small we can make the lines containing $f$ and $f'$ intersect arbitrarily close to

$[x_1, x_2]$, i.e. inside $V$. This is impossible – fibers cannot intersect. Therefore, we can choose $y_1$ and $y_2$ in such a way that the only fibers in $p$ intersecting the quadrangle $y_1\ x_1\ x_2\ y_2$ intersect $[x_1, x_2]$. The mapping $T = (P|_{N'})^{-1} \circ P$ maps this quadrangle to the plane $p'$. Consider the fiber going through $y_1$. If $y_1$ is sufficiently close to $x_1$, the angle between the fiber going through $y_1$ and the plane $p$ is close to the angle between $f_1$ and $p$.

The plane $p$ subdivides the space into two half-spaces $H$ and $H'$. Let $H$ be the half-space that contains part of the fiber $f_1$ whose projection into the plane $p$ lies in the same half-plane with respect to $(x_1, x_2)$ as $y_1$ and $y_2$. By our choice of $p$ the projection of $f_2 \cup H$ lies in the other half-plane.

Therefore, the projection into $p$ of the part of $F(y_1)$ which lies in $H$ will be close to the projection of $f_1$. By choosing $y_1$ close enough to $[x_1, x_2]$, we can guarantee that it lies entirely in the same half-plane of $p$ as the projection of $f_1$, and doesn't intersect $[x_1, x_2]$. As $p' \perp p$, $F(y_2) \cap p' = T(y_2)$ projects to a point on $[x_1, x_2]$. We conclude that $T(y_1)$ is not in $H$ and is, therefore, in $H'$. In the same way we can show that $T(y_2)$ is in $H$. Thus, $T(y_1)$ and $T(y_2)$ are in different half-planes of $p'$ with respect to $(x_1, x_2)$. $T([y_1, y_2])$ intersects $[x_1, x_2]$ at the point $x$, as the fiber $F(x)$ lies in $p$ and doesn't belong to $(x_1, x_2)$. If there is a path inside the image which goes from a point of the image on one side of the line $(x_1, x_2)$ to a point on the other side, there should be a point $z \in [x_1, x_2]$ on the path, because no point in $x_1\ x_2\ y_2\ y_1$ is mapped to a point of $(x_1, x_2)$ outside $[x_1, x_2]$. Therefore, the set $(x_1\ x_2\ y_2\ y_1)$ is subdivided into two disconnected parts by $[x_1, x_2]$.

Let $D$ be an open neighborhood of $T(x_1\ x_2\ y_2\ y_1)$ with $(x_1, x_2) \setminus [x_1, x_2]$ excluded. It has the same property with respect to $[x_1, x_2]$, and $T(y_1, y_2)$ is contained in the interior of $D$. As $F(x_1) \not\subset p$ and $F(x_2) \not\subset p$, there are neighborhoods of the endpoints of $[x_1, x_2]$ in $(x_1, x_2)$ that don't contain any points of of fibers that lie entirely in $p$. Therefore, we can pick two points $x'_1$ and $x'_2$ on $[x_1, x_2]$ that don't coincide with $x_1$ or $x_2$ and such that for all $x \in [x_1, x_2] \setminus [x'_1, x'_2]$ $F(x) \not\subset p$.

Define $D' = D \setminus ([x_1, x_2] \setminus (x'_1, x'_2)$. $D'$ has the same property as $D$. Consider perturbations of $[x_1, x_2]$ and $T(y_1, y_2)$. Sufficiently small perturbation doesn't move $[x'_1, x'_2]$ out of a band around $(x_1, x_2)$. It also doesn't move $x_1$ and $x_2$ into $D'$, and the endpoints of $T(y_1, y_2)$ out of their neighborhoods that have empty intersection with the band $T(y_1, y_2)$. Then perturbed $[x_1, x_2]$ retains the dividing property for $D'$ and the endpoints of $T(y_1, y_2)$ are still in different sets. Therefore, the intersection is preserved under small perturbations and is a crossing. This contradicts Condition T$'$. We conclude that the fibers should be coplanar.

This is sufficient (Lemma 3.3 ) to prove that the fibers of the projection $P$ coincide with the fibers of a perspective projection.

### 3.2.5 Decomposition

We have proved in previous sections that various sets of structural requirements result in the following property for projections:

**Theorem 3.1** *Given Condition H from Section 3.2.2, or Conditions L and T from Section 3.2.3, or Conditions $L'$ and $T'$ from Section 3.2.4, a projection $P$ satisfying these conditions should have fibers that are subsets of the fibers of a perspective projection* $\Pi$.

From this theorem it immediately follows that we can decompose the projection $P$ in two ways, described in the following

**Corollary 3.2** *Assume that the conditions of Theorem 3.1 hold. Let $\Pi$ be the perspective projection coresponding to $P$.*

1. $\Pi$ *is a central projection with center $C$. Assume, in addition, that the region $V$ lies entirely in one half-space with respect to a plane going through the center of* $\Pi$. *Then $P$ can be decomposed in two ways:*

   *– as a composition of a central projection $\Pi_{plane}$ into a plane followed by a smooth one-to-one transformation $T_{plane}$ of the plane, or*

   *– as a central projection $\Pi_{sphere}$ into a sphere with center $C$ followed by one-to-one smooth mapping $T_{sphere}$ of the sphere into the picture plane.*

2. $\Pi$ *is a parallel projection. Then $P$ can be decomposed as a composition of an orthogonal central projection $\Pi_{plane}$ into a plane followed by a smooth one-to-one transformation $T_{plane}$ of the plane.*

It is important to point out that the theorems were proven for a path-connected domain $V$. Therefore, if all the objects in a picture that are subject to the structural conditions described above can be separated into sets with each set lying in a separate path-connected domain (for example, foreground, middleground and background) a different perspective projection can be chosen in the decomposition for each set. It is interesting to see actual realization of similar principles in some Western and Japanese paintings (Section 2.6, Section 4.1).

## 3.3 Nonstructural Requirements

As we have mentioned in the beginning of this chapter, we will use nonstructural requirements to choose projections in the class defined by the structural requirements. Due to the more quantitative nature of these requirements, the appropriate way to formalize them is different. Rather than defining which images satisfy the requirement and which images don't, we define *error functions* that give us a measure of a nonstructural distortion. Formally speaking, these are functions on the set of all possible images. Functions of this type are difficult to deal with. Therefore, it will be more convenient to define error functions on the point of the picture plane, in such a way that the error function at each point gives an estimate of the maximal distortion of a small image at this point.

In this section we consider only projections on a path-connected open domain $V$ satisfying Theorem 3.1. Therefore, for each projection we can define the center of projection (unless all fibers are parallel) and decompositions as described in Section 3.2.5. We will assume that the fiber structure of the projection is fixed, i.e. projections $\Pi_{sphere}$ and $\Pi_{plane}$ in the decompositons are fixed.

### 3.3.1 Definitions

Let's establish some notation for the projections that satisfy the conditions of the theorem.

$$\begin{tikzcd} V \subset \mathbf{R}^3 \arrow[r, "\Pi_{plane}"] \arrow[d, "\Pi_{sphere}"'] \arrow[dr, "P"] & W_{plane} \subset \mathbf{R}^2 = p \arrow[d, "T_{plane}"] \\ W_{sphere} \subset \mathbf{S}^2 \arrow[r, "T_{sphere}"] & \mathbf{R}^2 = \pi \end{tikzcd}$$

We will consider projections $P : V \subset \mathbf{R}^3 \rightarrow \mathbf{R}^2$, from an open connected domain $V$ into the *picture plane* $\pi$, which are compositions of the projection $\Pi_{plane}$ from the center $O$ into the *intermediate plane* $p$ and a mapping $T_{plane} : W_{plane} \subset \mathbf{R}^2 \rightarrow \mathbf{R}^2$, $W_{plane} = \Pi_{plane}(V)$, We can choose the plane $p$ so that that the distance from $O$ to the plane is $L$.

$P$ can be also represented as a composition of central projection from $O$ into the sphere of radius $L$ with center at $O$ (*intermediate sphere* ) $\Pi_{sphere} : V \subset \mathbf{R}^3 \rightarrow \mathbf{S}^2$ and some mapping

$\mathsf{T} : W_{sphere} \subset \mathbf{S}^2 \to \mathbf{R}^2$, $W_{sphere} = \Pi_{sphere}(V)$, We will assume that the image of $V$ in the sphere belongs to a hemisphere.

If the fiber structure of the projection $P$ is that of a parallel projection, we will use only the first decomposition.

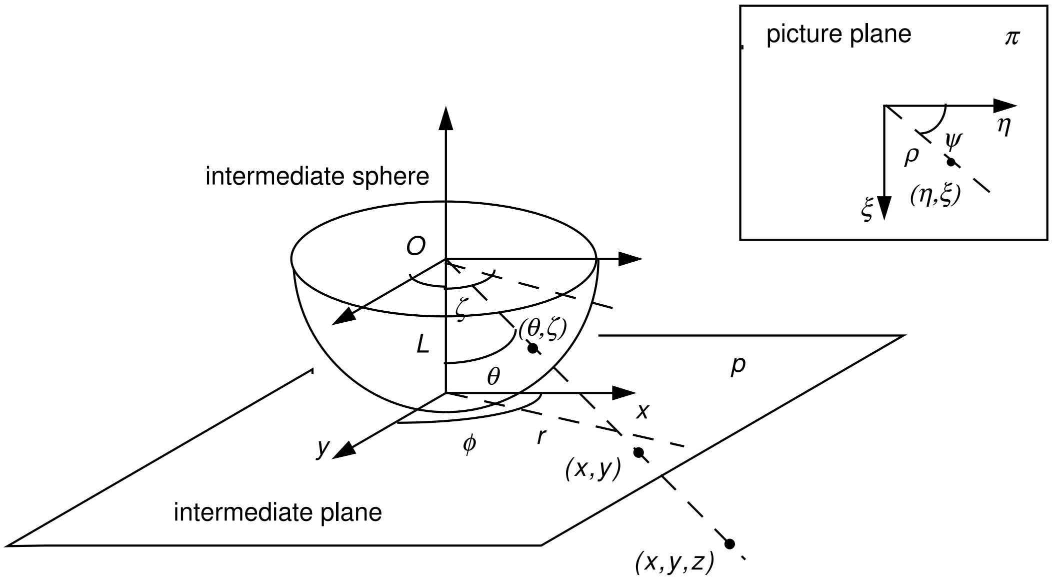


Figure 3.9: Coordinate systems. □

Let's introduce rectangular coordinates (x,y) and polar coordinates $(r, \phi)$ on the plane $p$; rectangular coordinates $(\eta, \xi)$ and polar coordinates $(\rho, \psi)$ in the plane $\pi$. On the sphere we will choose angular coordinate system $(\theta, \zeta)$, and local coordinates in the neighborhood of a fixed point $(\theta_0, \zeta_0)$: $u = L(\theta - \theta_0), v = L(\zeta - \zeta_0)\sin\theta_0$ (Figure 3.9).

The correspondence between $T_{plane}$ and $\mathsf{T}$ is given by the stereographic projection $\mathbf{S}^2 \to \mathbf{R}^2$: $\phi = \zeta$, $r = \tan\theta$.

### 3.3.2 Zero Curvature Condition

The requirement from Section 2.4.2 is perhaps the easiest to formalize:

> **Zero-curvature condition.** Images of line segments should have zero curvature at each point.

It would be desirable to use a measure of deviation from this condition similar to Watt's measure (Section 2.4.2). Watt's measure, however, is defined for a particular viewing position, while our measure has to be independent of the viewing conditions. A natural candidate for such measure is the curvature of the image of a straight line at a point.

As the curvature has the dimension of inverse length, we need a scaling factor to make the measure independent of the size of the picture. As a scaling factor we can choose some characteristic length. The most appropriate choice appears to be either a typical size of a line segment that can be found in a picture, or the size of the picture. The later choice gives a conservative measure, assuming the worst case: there are images of straight lines crossing the whole field of view.

Let's qualitatively compare this measure with Watt's measure. Watt's measure depends on the viewing conditions, and therefore, we have to make assumptions about these conditions. Due to the spatial limits of acute vision described in Section 2.4.1, we can assume that the picture occupies no more than $40°$ of the field of view. Most pictures are viewed from a distance when their angular size is not less then $5°$. Given these estimates, we have the viewing distance $l$ equal roughly to 4 to 20 times the size of the picture. Let $l = k_1 R_0$, where $R_0$ is the size of the picture. The area between a chord of length $d$ and a segment of a curve with radius of curvature $r$ is to the first order of approximation $d^3/6r$. Assuming $d$ to be of the order of magnitude of the size of the picture, $d = k_2 R_0$, where $k_2 < 1$ and close to one by the order of magnitude. The Watt's functional can be written as $C(k_2^3/6k_1^2)R_0/r$, where $C$ is less than unity, because we used a chord instead of a least-squares lines. Therefore, it is roughly proportional to the curvature times the size of the picture, with the coefficient of the order $10^{-2}$.

If we consider the decomposition $P = T_{plane} \circ \Pi_{plane}$, we can observe that $\Pi_{plane}$ satisfies the zero-curvature condition. Therefore, we have to consider only the mapping $T_{plane}$. We will denote components of $T_{plane}(x, y)$, which is a point in the picture plane, by $(\eta(x, y), \xi(x, y))$

The curvature depends not only on the point but also on the direction of the line whose image we are considering. As an error function for the zero-curvature condition at a point $x$ we will use an estimate of the maximum curvature of the image of a line going through $x$.

Let's estimate the maximum curvature of the image of a line defined by $\mathbf{x}+\mathbf{v}t$ in the intermediate projection plane, in the point $T_{plane}(\mathbf{x})$.

By definition, $\kappa = \frac{d\alpha}{ds}$, where $d\alpha$ is the change in the direction of the tangent vector of the curve along the segment $ds$.

Let $\mathbf{w}$ be the tangent of the curve $\gamma = \gamma(t)$: $\mathbf{w} = \frac{d\gamma}{dt}$. Then

$$|\kappa| = \frac{\left|\mathbf{w} \times \frac{d\mathbf{w}}{dt}\right|}{|\mathbf{w}|^3} \le \frac{|\mathbf{w}|\left|\frac{d\mathbf{w}}{dt}\right|}{|\mathbf{w}|^3} = \frac{\left|\frac{d\mathbf{w}}{dt}\right|}{|\mathbf{w}|^2}$$

In our case the curve is defined by the equation $\gamma(t) = T_{plane}(\mathbf{x}+\mathbf{v}t)$. To estimate the curvature from above, we will estimate the maximum of $\frac{\left|\frac{d\mathbf{w}}{dt}\right|}{|\mathbf{v}|^2}$ and the minimum of $\frac{|\mathbf{w}|^2}{|\mathbf{v}|^2}$ for a fixed point $\mathbf{x}$. Using $\frac{d}{dt}F(\mathbf{x}+\mathbf{v}t) = (\nabla F, \mathbf{v})$ for a scalar function $F$, we can write:

$$\begin{aligned}
\left|\frac{d\mathbf{w}(t)}{dt}\right|^2 &= \left|\frac{dw_\eta}{dt}\right|^2 + \left|\frac{dv_\xi}{dt}\right|^2 = (\nabla w_\eta, \mathbf{v})^2 + (\nabla w_\xi, \mathbf{v})^2 \\
&\le (|\nabla w_\eta^2 + |\nabla w_\xi|^2)|\mathbf{v}|^2 = (|\nabla \frac{d\eta}{dt}|^2 + |\nabla \frac{d\xi}{dt}|^2)|\mathbf{v}|^2 \\
&= (|\nabla(\nabla\eta, \mathbf{v})|^2 + |\nabla(\nabla\xi, \mathbf{v})|^2)|\mathbf{v}|^2 = (|(\mathbf{v}, \nabla)\nabla\eta|^2 + |(\mathbf{v}, \nabla)\nabla\xi|^2)|\mathbf{v}|^2 \\
&\le (|(\mathbf{e}_x, \nabla)\nabla\eta|^2 + |(\mathbf{e}_y, \nabla)\nabla\eta|^2 + (\mathbf{e}_x, \nabla)\nabla\xi|^2 + (\mathbf{e}_y, \nabla)\nabla\xi|^2)|\mathbf{v}|^4
\end{aligned}$$

Expanding $|(\mathbf{e}_x, \nabla)\nabla\eta| = |\eta_{xx}|^2 + |\eta_{xy}|^2$ and other terms in similar way, we get

$$\left|\frac{d\mathbf{w}(t)}{dt}\right| \le \sqrt{|\eta_{xx}|^2 + |\eta_{yy}|^2 + 2\,|\eta_{xy}|^2 + |\xi_{xx}|^2 + |\xi_{yy}|^2 + 2\,|\xi_{xy}|^2}|\mathbf{v}|^2$$

Now let's estimate $|\mathbf{w}|^2$ from below.

$$\begin{aligned}
|\mathbf{w}|^2 &= (\nabla\eta, v)^2 + (\nabla\xi, v)^2 \\
&\le ((\eta_x^2 + \xi_x^2)v_x^2 + 2(\eta_x\eta_y + \xi_x\xi_y)v_x v_y + (\eta_y^2 + \xi_y^2)v_y^2) \\
&= Av_x^2 + 2Bv_y v_x + Cv_y^2
\end{aligned}$$

We can find $\alpha$ such that

$$\cos\alpha = \frac{w_u}{\sqrt{w_u^2 + w_v^2}}, \quad \sin\alpha = \frac{w_v}{\sqrt{w_u^2 + w_v^2}},$$

Thus

$$|\mathbf{w}|^2 = A\cos^2\alpha + 2B\cos\alpha\sin\alpha + C\sin^2\alpha$$

.

Minimizing this expression w.r.t. $\alpha$, we get

$$|\mathbf{w}|^2 \geq \frac{1}{2}((A+C) - \sqrt{(A-C)^2 + 4B^2})|\mathbf{v}|^2$$

Finally, we get the estimate for the curvature:

$$|\kappa| \leq \frac{\sqrt{|\eta_{xx}|^2 + |\eta_{yy}|^2 + 2\,|\eta_{xy}|^2 + |\xi_{xx}|^2 + |\xi_{yy}|^2 + 2\,|\xi_{xy}|^2}}{\frac{1}{2}((A+C) - \sqrt{(A-C)^2 + 4B^2})}$$

To simplify calculations, we will use the square of the curvature:

$$K(T_{plane}, x, y) = \frac{|\eta_{xx}|^2 + |\eta_{yy}|^2 + 2\,|\eta_{xy}|^2 + |\xi_{xx}|^2 + |\xi_{yy}|^2 + 2\,|\xi_{xy}|^2}{\frac{1}{4}((A+C) - \sqrt{(A-C)^2 + 4B^2})^2} \tag{3.1}$$

If we set $K(T_{plane}, x, y) = 0$, we see that all of the second derivatives of $\eta$ and $\xi$ should be equal to zero, and so $T_{plane}$ should be a linear transformation. This coincides with the "main theorem of affine geometry", which says that the only transformations of the plane which map lines into lines are linear transformations.

### 3.3.3 Direct View Condition

Perceptual distortions of the shape of rectangular corners, spheres, bodies, cylinders, etc. (Section 2.3) don't occur when we use narrow-angle perspective projection, i.e. when the principal ray goes through the object and the angular subtense of the object when viewed from the projection point is small. Therefore, the following condition can be considered a generalization of a number of nonstructural perceptual requirements:

> **Direct-view condition.** It is desirable for the projection to be close in a neighborhood of each point to the perspective projection into a plane perpendicular to the fiber line going through this point.

First, we consider the case when the fiber structure of the mapping coincides with the fiber structure of a central projection.

We can observe that the projection $\Pi_{sphere}$ into the sphere is locally close to the projection into the tangent plane of the sphere which is perpendicular to the fiber lines of $P$.

Therefore, if we use the decomposition $P = \mathsf{T} \circ \Pi_{sphere}$ we have to construct the mapping $T_{sphere}$ which is locally as close to a similarity mapping as possible. Formally, it means that the differential of the mapping $T_{sphere}$, which maps the tangent plane of the sphere at each point $x$ to the plane $T_{f(x)}\mathbf{R}^2 = \mathbf{R}^2$ coinciding with the picture plane $\pi$ at the point $f(x)$, should be close to a similarity mapping. The differential $DT_{f(x)}$ can be represented by the Jacobian matrix $J$ of the mapping $T_{sphere}$ at the point $x$.

In coordinate form this can be written as

$$J_{f,x} = \alpha^2 I \tag{3.2}$$

for some $\alpha \neq 0$.

For local coordinates $(u, v)$ on the sphere the matrix $J_{f,x}$ can be written as

$$J_{f,x} = \begin{pmatrix} \eta_u & \eta_v \\ \xi_u & \xi_v \end{pmatrix}$$

Then for ( 3.2) to be satisfied it is necessary and sufficient that

$$|J_{f,x}| \neq 0, \quad \eta_u = \xi_v, \quad \eta_v = -\xi_u$$

Note that these equations formally coincide with Cauchy-Riemann conditions. If we consider the sphere and the plane to be complex manifolds, then the equations above are necessary and sufficient conditions for $T_{sphere}$ to be a conformal mapping.

For global spherical coordinates we have

$$\eta_u = \eta_\theta, \qquad \eta_v = \eta_\zeta \frac{1}{\sin\theta} \tag{3.3}$$

$$\xi_u = \xi_\theta, \qquad \xi_v = \xi_\zeta \frac{1}{\sin\theta} \tag{3.4}$$

We will say that mappings $T_{sphere}$ and $P$ satisfy direct view condition, if they satisfy the conditions 3.3.

We also need of measure of deviation from the direct view condition. A nondegenerate linear transformation $J$ is a similarity transformation if and only if $|Jw|/|w|$ doesn't depend on $w$. If this ratio depends on $w$, then we define the direct view error function to be

$$D(T_{sphere}, \theta, \zeta) = \left| \max\left(\frac{|Jw|^2}{|w|^2}\right) / \min\left(\frac{|Jw|^2}{|w|^2}\right) - 1 \right| \tag{3.5}$$

$D(T_{sphere}, \theta, \zeta)$ can be used as the measure of "non-directness" of the transformation at the point.

This measure has a simple geometric meaning. Suppose we have a small sphere at some point in space. The image of the sphere on the intermediate sphere is a small disk. Then this disk is mapped by $T_{sphere}$ into the picture plane. Ideally, this image should be a circle, and as it was noted in Section 2.3.2, the aspect ratio of the image is a reasonable measure of distortion in this case. The functional that we have defined is the deviation from 1 of the aspect ratio of the image of a small sphere. This measure can be also related to the maximal offending angle in the Perkins' laws (Section 2.3.1).

Let's find the explicit expression for $\min \frac{|Jw|^2}{|w|^2}$ and $\max \frac{|Jw|^2}{|w|^2}$. We will use the following notation:

$$\begin{aligned} E &= (\eta_u)^2 + (\xi_u)^2, \\ F &= \eta_u \eta_v + \xi_u \xi_v, \\ G &= (\eta_v)^2 + (\xi_v)^2. \end{aligned}$$

Then $|Jw|^2 = Ew_u^2 + 2Fw_v + Gw_v^2$.

This expression is identical to the expression in Section 3.3.2 so we immediately get

$$\begin{aligned} \min \frac{|Jw|^2}{|w|^2} &= \frac{1}{2}((E+G) - \sqrt{(E-G)^2 + 4F^2}) \\ \max \frac{|Jw|^2}{|w|^2} &= \frac{1}{2}((E+G) + \sqrt{(E-G)^2 + 4F^2}) \end{aligned}$$

Finally, for the direct projection error function $D$ we get

$$D(T_{sphere}, \theta, \zeta) = \frac{(E+G) - \sqrt{(E-G)^2 + 4F^2}}{(E+G) + \sqrt{(E-G)^2 + 4F^2}} - 1 \qquad (3.6)$$

Using the correspondence between the intermediate sphere and the plane we can write $D$ as a function of $T_{plane}$, $x$ and $y$.

The only transformations that satisfy the direct view condition exactly are the conformal mappings from the sphere to the plane.

As it can be easily shown, if $T_{plane}$ is linear, corresponding $T_{sphere}$ is not conformal.

Now we consider the case of projections $P$ with parallel fibers. In this case, the projection to the intermediate plane has the direct view property. All conformal transformations of the plane will preserve the direct view property. Then the conditions 3.3 become standard Cauchy-Riemann equations.

The same measure of deviation from the direct view condition (3.5) can be used, but in this case we take the local coordinates $(u, v)$ to be the same as $(\eta, \xi)$.

In this case any composition of orthogonal and similarity transformation of the plane will be conformal and linear and will satisfy the direct view condition. But any projection $T_{plane} \circ \Pi_{plane}$ where $T_{plane}$ is a composition of orthogonal and similarity transformation is equivalent to an orthogonal parallel projection up to a scaling factor. Therefore, we arrive at an important conclusion:

> The direct view condition can be satisfied simultaneously with the zero-curvature condition within the class of projections described by Theorem 3.1 only if the projection coincides with an orthogonal parallel projection to the plane up to a similarity transformation of the plane.

### 3.3.4 Error Functionals

In view of the statement in the previous subsection, we can see that only orthogonal parallel projection satisfies both the direct-view and zero-curvature conditions simultaneously. It is quite obvious that parallel projection is not a satisfactory choice in many cases. In Section 5.2 we examine some reasons for that. Therefore, we are also interested in projections that have the fiber structure of a central projection.

Since for central projections the direct-view and zero-curvature conditions cannot be satisfied exactly, we should find a way to define the optimal tradeoff between these two types of distortions. In order to do this, it is convenient to use global error functionals which indicate the quality of the projection with respect to a given condition.

In order to estimate the total error for a projection, we have to choose a measure and integrate the error function over the domain of the $T_{plane}$ (we will assume that the direct projection error function is converted to the intermediate projection plane coordinates: $D = D(T_{plane}, x, y)$.) The most conservative way to define the curvature and direct projection functionals is to use $L_\infty$ measure, which results, for continuous $D$ and $K$, in

$$\mathcal{K}[T_{plane}] = sup_{x,y} K(T_{plane}, x, y) \quad \mathcal{D}[T_{plane}] = sup_{x,y} D(T_{plane}, x, y). \tag{3.7}$$

Other possibilities are, for example, $L^p$ measures:

$$\mathcal{K}[T_{plane}] = \left( \int \int_{x,y} K(T_{plane}, x, y)^p dxdy \right)^{\frac{1}{p}}, \mathcal{D}[T_{plane}] = \left( \int \int_{x,y} D(T_{plane}, x, y)^p dxdy \right)^{\frac{1}{p}}. \tag{3.8}$$

### 3.3.5 Optimization Problem

Initially we don't make any assumptions about the form of the functional that we choose, except that it should be some functional norm in the space of smooth functions. Assuming $T_{plane}$ to be smooth, $\mathcal{K}[T_{plane}] = 0$ if and only if $K(T_{plane}, x, y) = 0$ almost everywhere. The same is true for $\mathcal{D}$. Then the only functions satisfying $K[T_{plane}] = 0$ are linear functions and the only functions satisfying $D[T_{plane}] = 0$ are those derived from conformal mappings of the sphere onto the plane. Therefore, we cannot make both functionals equal to zero for any choice of the norm. So we want for a given value of one functional to find a function $T_{plane}$ such that the other functional has the smallest possible value.

$$\text{minimize} \quad \mathcal{K}[T_{plane}] = \nu, \text{ subject to } \quad \mathcal{D}[T_{plane}] = \mu \tag{3.9}$$

or,

$$\text{minimize} \quad \mathcal{D}[T_{plane}] = \mu, \text{ subject to } \quad \mathcal{K}[T_{plane}] = \nu \tag{3.10}$$

We want to find a family of functions $T_{plane}(\mu)$ or $T_{plane}(\nu)$ solving the constrained optimization problems stated above. In each case we get a function $\nu(\mu)$ or $\mu(\nu)$.

We can reduce the constrained problem stated above to an unconstrained one in the following way:

Consider a linear combination of the functionals $\mathcal{F}[T_{plane}] = \lambda\mathcal{K}[T_{plane}] + (1-\lambda)\mathcal{D}[T_{plane}])$. If $T_{plane_0}$ minimizes $\mathcal{F}[T_{plane}]$ for a fixed $\lambda$, then $\mathcal{K}[T_{plane_0}]$ is minimal for the fixed value of $\mathcal{D}[T_{plane}] = \mathcal{D}[T_{plane_0}]$. As we change $\lambda$, the value of $\mathcal{D}[T_{plane_0}]$ will increase up to the maximal value, which is attained when $\lambda = 1$. Therefore, instead of minimizing $\mathcal{K}[T_{plane}]$ for fixed values of $\mathcal{D}[T_{plane}]$, we can minimize the functional $\mathcal{F}[T_{plane}]$ for $\lambda$ varying from 0 to 1. The same is true for minimizing $\mathcal{D}[\mathsf{T}]$ for fixed values of $\mathcal{K}[T_{plane}]$. The problem can be restated as follows:

$$\text{minimize} \quad \mathsf{F} = \lambda\mathcal{D} + (1-\lambda)\mathcal{K} \quad \text{for } \lambda \in [0,1]$$

We also have to specify the boundary conditions in order to make the problem well-defined. This can be done by fixing the frame of the picture, that is, the values of $T_{plane}$ on the boundary of $W$.

To ensure that the mapping is one-to-one and the inverse is smooth, the Jacobian of the mapping should not be zero anywhere.

It would also be desirable not to increase one distortion *anywhere* in the image beyond the level that is possible when the other distortion is 0. It is a somewhat restrictive requirement and potentially can be relaxed if it is possible to obtain considerable reductions in maximal distortion at the expense of a small increase of of small values of distortion. It is unlikely for the distortion of curvature — if the curvature is 0 in some part of the image, small increases in the curvature are likely to be observable, because of the threshold nature of the perception of straightness [Watt et al., 1987]. It is potentially possible to gain some improvement by relaxing the same requirement for the direct view condition: small distortions of shape are undetectable for most objects.

Mathematically the restrictions above can be formulated as

$$K(x,y) < K_{conf}(x,y), \quad D(x,y) < D_{lin}(x,y),$$

where $K_{conf}(x,y)$ is the value of the error function at the point $(x,y)$ for a conformal mapping $(D(x,y) \equiv 0)$, and $D_{lin}(x,y)$ is the value of the error function for the identity mapping (the only one which has $K(x,y) \equiv 0$.) $K_{conf}(x,y)$ depends on the choice of the conformal mapping – this choice can be arbitrary, because all conformal mappings result in $D(x,y) \equiv 0$, and any is sufficient if condition (3.3.5) holds.

In Section 4.3, we will explore a particular case of this optimization problem for $T_{plane}$ with central symmetry.

It is also interesting to consider weighted error functions, that is, $w_K(x,y)K(T_{plane},x,y)$ and $w_D(x,y)D(T_{plane},x,y)$, where $w_K(x,y)$ and $w_D(x,y)$ are nonnegative weights.

The weights of the error functions can be interpreted as the relative importance of a particular type of perceptual error close to the given point in the image. It can be user-specified or calculated using information about the object space.

In this case the total error can be defined as the norm $||w_K(x,y)K(T_{plane},x,y) + w_D(x,y)D(T_{plane},x,y)||$, and the optimal transformation.

$K - D$ **dependencies.** As we have shown, the original optimization problem can be reduced to minimization of one functional $\mathsf{F}$. This functional doesn't have a good perceptual interpretation. A good way to characterize the optimal family of transformation is by the $(\mathcal{K},\mathcal{D})$ pairs corresponding to each member of the family.

If we solve the optimization problem (3.3.5), we get a one-parametric family parametrized by $\lambda$, which defines a parametric curve $(\mathcal{K}(\lambda),\mathcal{D}(\lambda))$. This curve allows to see to what extent distortions can be eliminated simultaneously.

It is necessary to emphasize that the optimization problem formulated in this section is not equivalent to the task of minimizing distortions. It is a specific formalization of this task with relatively low dependence on the scene contents. Projections can be improved perceptually by introducing more dependence on scene contents. The tradoff is between the difficulty of the construction of an appropriate projection and the quality of the resulting image. We tried to choose a path in between, in which the choice of projections is limited, but some improvement can still be achieved.