Chapter 4

Construction of Projections

4.1 Previous Work

A number of reasons lead to the constructions of picture projections that were different from the linear perspective. There were three main motivations: perceptual [Raushenbakh, 1986], [Reggini, 1975]), [Stark, 1928], artistic ([Ernst, 1976], [Inakage, 1991], [Flocon and Taton, 1963], and technical [Goncharenko et al., 1979] considerations.

4.2 Perceptually-based systems

Reggini's perspective. Reggini's system of perspective [Reggini, 1975] is based on Thouless' formula ([Thouless, 1931a], [Thouless, 1931b], [Thouless, 1932]), rewritten as

$$\frac{p}{r} = \left(\frac{t}{d}\right)^{1-i}, \tag{4.1}$$

where t is the distance from the projection point to the object, d is the distance from the projection point to the plane and i is a constant between 0 and 1. The projection of a point is computed in the following way: let r be the distance from the point to the principal ray (the ray through the projection point perpendicular to the projection plane.) Let t be the distance from the point to the projection plane. A point X in three-dimensional space can be specified by three numbers (r, t, ϕ) in the cylindrical coordinate system with the origin coinciding with the projection point and the axis coinciding with the principal ray.

In this system the rays (sets of points mapping to a single point in the projection plane) are curved. In the previous chapter we have discussed the problems that are associated with this curvature.

The main goal of Reggini's system is to provide for changing parallel foreshortening; as a result, close objects have smaller degree of parallel foreshortening. The effective viewpoint, determined by the intersection point of the tangent lines to the rays at a given distance from the picture plane, moves away from the projection plane as the distance to the object increases.

Raushenbakh's system. Raushenbakh's system [Raushenbakh, 1986] is based on an idea similar to Reggini's. Instead of using Thouless' formula, he derives similar curvilinear systems under more general assumptions about the relationship of the apparent size on the projected size. More specifically, Raushenbakh assumes that the apparent size is related to the projected size by the formula

$$s^* = sF(D)$$

where s^* is the apparent size, s is the size of projection onto a fixed plane, D is the distance from the plane to the object. He suggest a technique for measuring $F(D)$.

The result of his derivations is a family of projections; different systems correspond to different tradeoffs between vertical and horizontal deviation of the depicted size from the apparent size.

Both systems that were described above are based on the assumption that the apparent size should be depicted. We should note, however, that the size constancy works also for pictures, although less than for the real scenes. Still there is a difference in the perception of size in pictures and in the real world, and some intermediate values between apparent and projected size might create the most appropriate impression. These systems are also useful for understanding some deviations from linear perspective in paintings and naive art – some of the representations of objects that are consistently present in the pictures, could result from unconscious attempts to depict the apparent size.

We believe that main problem of these systems is that the distortion associated with relative size can be found mostly in wide-angle images, and a number of other distortions , such as the distortion of shape, in these image tend to be more prominent. The systems described above might result in reduction of these distortions but nothing in their construction guarantees that. The important problem of the curvature of images of straight lines is not addressed either.

We don't describe two more systems - Flocon's and Stark's - because we didn't have the original references available and we could not find satisfactory accounts of their work.

4.2.1 Artistic systems

In a sense, any projection system falls into this category, because any system can be used for artistic purposes. We describe several systems that were created specifically for artistic purposes.

Escher's perspective. M. C. Escher is famous for geometric construction in his paintings. He used unuasual perspective constructions in many works, and one of his experiments (Figure 4.1) is particularly relevant to our discussion. From Escher's notes that he had made when he was working on this woodcut, we know that he was trying to create a smooth transition between different views of a building - with the viewing direction changing from horizontal to almost vertical. Resulting construction is similar to the panoramic projection described below and allows to avoid distortion of rectangular corners in all parts of this complicated construction.

Systems described by M. Inakage. A number of projections were described in [Inakage, 1991].

Inakage defines *curvilinear perspective* as a linear projection of the three-dimensional space into a curved surface. This definition implies that the surface is projected into the plane to get the final image. Therefore, this definition is equivalent to the decomposition given by Theorem 3.1.

3D-warping extends the curvilinear perspective, allowing to change the metrics of space, producing curved rays. Any continuous system can be considered an instance of curvilinear perspective combined with 3D-warping.

Specific case of *inverse perspective* is described by the formulas

$$x' = x(\frac{z}{d} + 1); \quad y' = y(\frac{z}{d} + 1), \tag{4.2}$$

Figure 4.1: M. C. Escher. High and Low,lithograph, 1947 □

where (x, y, z) is a point in space (x', y') is it's projection and d is the distance from the "projection point" to the projection plane.

Another example of perspective constructions are the photographic collages by David Hockney [Hockney, 1988].

As we have mentioned in Chapter 2, deviations from linear perspective in the art are probably more common than rigorous perspective constructions. Rejection of linear perspective was one of the theoretical foundations of cubism. The systems that we have described above are just several examples of attempts to use alternative methods of projection.

4.2.2 Technical Projections

Extreme distortions of linear perspective for angles close to 180°, and impossibility of using it for projection angles greater than 180°, prompted several practical solutions, primarily in lens design. For extremely wide-angles (greater than 110-120 degrees,) linear projection is never used because of the distortions and difficulties in creating appropriate rectilinear lenses. Fish-eye lenses are based on projections that are different from linear perspective. All of these projections have straight rays (unless the light propagates in a medium with variable refraction coefficient.)

Several projections used in lens design are described in [Goncharenko et al., 1979]. The fiber structure of all these projections is identical to the central projection –all of the fibers are straight lines intersecting at the center of the lens, which can be called the projection point. Moreover, these projections have axial symmetry. Let p be the axis of symmetry, which corresponds to the ray going through the center of the lens and its focal points. Let β be the angle between the direction from the projection point to a point in space X. We will use a polar coordinate system (r, ϕ) on the projection plane with the origin at the intersection point of the axis and the plane. Then the image of the point X is determined by a function $r = F(\beta)$.

Three projections described in [Goncharenko et al., 1979] are shown in the table below (f is the focal length of the lens.)

name	formula	examples of lenses
orthographic	$r = f \sin \beta$	Fisheye-Nikkor OP (180°)
equidistant	$r = f\beta$	Kowa Fisheye (180°), Olympus Zuiko-Fisheye (180°), Nikkor-Fisheye (220°)
equalized	$r = 2f \sin \frac{\beta}{2}$	Asahi Super Takumar-Fisheye, Minolta MC Fish-eye

Another example of a projection that is widely used in photography is the panoramic projection. If the film in a panoramic camera is placed on a cylindrical surface, the resulting projection resembles equidistant projection in one direction and regular linear perspective in the other. Panoramic projection was advocated by [Feininger, 1953] as the only "correct" perspective, on the grounds that parallel lines should always be seen as convergent to a point at infinity.

The projections that we have described in this section were mostly *ad hoc* solutions to the problems of wide-angle images. No consistent effort was made to minimize geometric distortions.

4.3 Axially Symmetric Projection

In general, the optimization problems formulated in Section 3.3.5 are very difficult. In this section we will consider the case of axially symmetric projections. The transformation T_{plane} for these projections can be written in polar coordinates as

$$\rho = \rho(r), \quad , \psi = \phi \tag{4.3}$$

In this case the problem can be reduced to a one-dimensional problem. Unfortunately, even in this form direct solution of the optimization problem is quite difficult. We shall make some estimates of the optimal values of the functional and show that some simple families of projections are close enough to optimal. We shall consider some applications of one of these families in Chapter 5.

4.3.1 Reformulation of the Problem

Using Equations 4.3 for the transformation T_{plane}, we can simplify the expressions for error functions. After substitutions, we obtain

$$K(\rho, r) = R^2 \frac{\frac{3}{r^2}\left(\frac{\rho}{r} - \rho'\right)^2 + \rho''^2}{\left(\frac{\rho^2}{r^2} + \rho'^2 - \left|\frac{\rho^2}{r^2} - \rho'^2\right|\right)^2} = R^2 \frac{\frac{3}{r^2}\left(\frac{\rho}{r} - \rho'\right)^2 + \rho''^2}{\min\left(\frac{\rho^2}{r^2}, \rho'^2\right)^2}$$

where R is the scaling constant.

The expression above is for the transformation $T_{plane} : \mathbf{R}^2 \to \mathbf{R}^2$. The direct projection functional was derived for $T_{sphere} : \mathbf{S}^2 \to \mathbf{R}^2$. The formulas relating T_{plane} and T_{sphere} are $\phi = \zeta,\ r = \tan\theta$.

The direct view functional has the form

$$\frac{\max(\frac{d\rho}{d\theta}^2, \frac{\rho^2}{\sin\theta})}{\min(\frac{d\rho}{d\theta}^2, \frac{\rho^2}{\sin\theta})} - 1$$

Using $r = \tan\theta$, we get

$$D(r,\rho) = \frac{\max(\rho'^2(1+r^2)^2, \rho^2(1+\frac{1}{r^2}))}{\min(\rho'^2(1+r^2)^2, \rho^2(1+\frac{1}{r^2}))} - 1$$

We can notice that in both cases the dependence on the angular coordinate completely disappeared, so now the problem is one-dimensional.

We should also note that if ρ doesn't depend on θ, the only consistent way to define the boundary conditions is to specify the value of ρ on a circle, i.e. for $r = R$, where R is the size of the picture (we will assume that we use it as the scaling factor in the curvature error function.)

As the transformation T_{plane} is required to be one-to-one and continuous everywhere, $\rho(0) = 0$ and $\rho'(r) \geq 0$ everywhere. In order for the error functions to be finite at 0, $\rho'(0) \neq 0$. In general, it is desirable to have $\rho'(r) > 0$ for all r, because otherwise we get an "infinite compression factor" at the points where $\rho' = 0$.

Moreover, if we want the transformation to be smooth at zero, $\rho''(0) = 0$.

The optimization problem for the *sup*-norm now takes the form

$$\text{minimize} \quad \mathsf{F}[\rho] = \max_{r\in[0,1]} \lambda K(\rho, \rho', \rho'', r), +(1-\lambda) D(\rho, \rho', r) \tag{4.4}$$

$$\text{subject to} \quad \rho(0) = 0,\ \rho(R) = R,\ \rho'(r) > 0, \rho''(0) = 0 \tag{4.5}$$

4.3.2 Limiting Cases

Consider the cases $\lambda = 1$ and $\lambda = 0$. In the first case the optimal solution is $\rho(r) = r$. In the second case we obtain a differential equation

$$\frac{\rho'}{\rho} r\sqrt{1+r^2} = 1$$

which can be easily solved. The solution satisfying $\rho(0) = 0,\ \rho(R) = R$ is

$$\rho(r) = \frac{R^2(\sqrt{1+r^2}-1)}{r(\sqrt{1+R^2}-1)} \tag{4.6}$$

The expression for the composition of the stereographic projection and ρ, which can be obtained by substituting $r = \tan\theta$ into the expression for ρ, is more intuitive:

$$\rho(\tan\theta) = C\tan\theta/2,$$

where C is a normalization coefficient, and results in the following expression for ρ:

$$\rho(r) = \frac{R\tan\frac{1}{2}\arctan r}{\tan\frac{1}{2}\arctan R}$$

This mapping has a simple geometric interpretation (Figure 4.2): first, the plane is projected to the intermediate unit sphere, and then reprojected back from the pole.

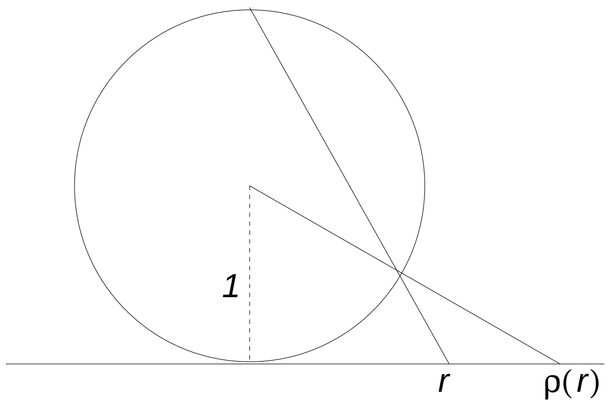

Figure 4.2: Construction of $DP(r)$ □

This mapping has an important role in the rest of the thesis. We will call it *direct view transformation* and denote it $DV(r, R)$.

In Figure 4.3 we compare this transformation with transformations implemented in the fisheye lenses.

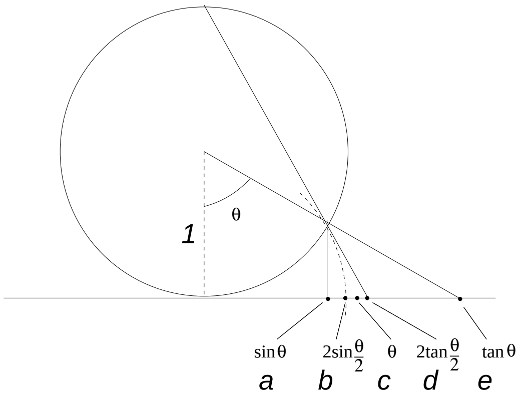

Figure 4.3: Projections: orthographic (a), equalized (b), equidistant (c), $DP(r)$ (d), linear (e). □

4.3.3 Lower Bound of the Optimal Values of F

In this section we will find a lower bound of the optimal values of the functional 4.4.

First, we observe that

$$\begin{aligned}&\min_{\rho} \mathsf{F}[\rho] = \min_{\rho} \max_{[0\ R]} F(\rho, \rho', \rho'', r) \geq \\ &\min_{\rho} F(\rho(R), \rho'(R), \rho''(R), R) = \min_{\rho} F(R, \rho'(R), \rho''(R), R) \geq \\ &\min_{x_1, x_2} F(R, x_1, x_2, R)\end{aligned}$$

Therefore, we can estimate the minimal value of the functional from below by minimizing the function

$$F(1, x_1, x_2, R) = \lambda \frac{\frac{3}{R^2}(1-x_1)^2 + x_2^2}{\min(1, x_1^2)^2} + (1-\lambda)\left(\max\left(\left(1+R^2\right) x_1^2, \frac{1}{\left(1+R^2\right) x_1^2}\right) - 1\right)$$

If $\lambda \neq 0$, x_2 should be zero at the point where the minimum is achieved. We can consider the function F of only one variable x_1. Assume $\lambda \neq 0$ (we shall consider the case $\lambda = 0$ case later.) Then we can divide the functional by λ and use $\alpha = (1-\lambda)/\lambda$.

Consider three ranges of x_1: $0 < x_1 \leq 1/\sqrt{R^2+1}$, $1/\sqrt{R^2+1} \leq x_1 \leq 1$, and $x_1 > 1$.

1. $0 \leq x_1 \leq 1/\sqrt{R^2+1}$. In this case the function can be rewritten as

$$F(1, x_1, x_2, R) = \frac{\frac{3}{R^2}\left(1-x_1\right)^2}{x_1^4} + \alpha\left(\frac{1}{\left(1+R^2\right)x_1^2} - 1\right) \tag{4.7}$$

   The derivative of the function is

$$F'(1, x_1, x_2, R) = \frac{-\frac{6}{R^2}\left(1-x_1\right)\left(2-x_1\right)}{x_1^5} - \frac{2\alpha}{\left(1+R^2\right)x_1^3} \tag{4.8}$$

   In this range of x_1 the derivative is clearly negative. Therefore, if the minimum is attained in this range, it must be attained at the right endpoint $1/\sqrt{1+R^2}$.

2. $x_1 > 1$. In this case the function takes the form

$$F(1, x_1, x_2, R) = \frac{3}{R^2}\left(1-x_1\right)^2 + \alpha\left(\left(1+R^2\right)x_1^2 - 1\right) \tag{4.9}$$

   The derivative of the function is

$$F'(1, x_1, x_2, R) = \frac{6}{R^2}(x_1 - 1) + 2\alpha\left(1 + R^2\right)x_1 \tag{4.10}$$

The derivative is positive in this range, therefore, the function is increasing and the minimum can be attained only at the left endpoint $x_1 = 1$.

3. $1/\sqrt{R^2+1} \leq x_1 \leq 1$. In this case the function and the derivative are

$$F(1, x_1, x_2, R) = \frac{\frac{3}{R^2}(1 - x_1)^2}{x_1^4} + \alpha\left(\left(1 + R^2\right)x_1^2 - 1\right) \tag{4.11}$$

$$F'(1, x_1, x_2, R) = \frac{-\frac{6}{R^2}(1 - x_1)(2 - x_1)}{x_1^5} + 2\alpha\left(1 + R^2\right)x_1 \tag{4.12}$$

At the right endpoint of the interval the derivative is always nonnegative; at the left endpoint it might be positive or negative.

The second derivative of the function in this range is

$$F''(1, x_1, x_2, R) = \frac{\frac{30}{R^2}(1-x_1)(2-x_1)}{x_1^6} + \frac{\frac{6}{R^2}(2-x_1)}{x_1^5} + \frac{\frac{6}{R^2}(1-x_1)}{x_1^5} + 2\alpha\left(1+R^2\right) \tag{4.13}$$

It can be easily seen that each term is positive inside the interval. Therefore, the first derivative increases inside the interval and has no more than one root. If it is positive at the left endpoint, there are no roots, otherwise, there is one root.

Using the continuity of the function, we conclude that the minimum is attained at a point in the interval $1/\sqrt{1+R^2} \leq x_1 \leq 1$ for any value of α.

If $\lambda = 0$, it is easy to see that in this case the minmal value of F is 1 and is attained at $x_1 = 1/\sqrt{1+R^2}$.

If the value of D at R is known, we can find $\rho'(r)$.

The $K - D$ curve for the bound is shown in Figure 4.4 for $R = 1$ (field of view 90°.)

4.3.4 Second Bound

The bound that we have derived in the previous section is not very tight, which it can be easily seen for $\lambda = 0$. In this case the minimum of the functional is 0, and the optimal solution is unique (Section 4.3.2.) The values of K in this case are much higher then the bound found in the previous section. In this section we shall make some additional assumptions about ρ and find a better bound for the optimal pairs (K, D) in a narrower class of functions. We believe that this bound is still far from the optimal, it is, however, the best we could obtain. Although the optimization problem that we consider can be easily reduced to a problem of optimal control in the form of Mayer with inequality constraints [Pontriagin, 1962], the necessary conditions that we could find don't yield a good estimate of the optimal values of the functional F, or of the optimal solution, in particular because the second derivative of the Hamiltonian with respect to the control vanishes identically. The elementry estimate that we derive in this section supports the idea that the interpolations derived in the next section are close to optimal but, of course, doesn't prove it.

We make the following assumptions about ρ:

1. D,K are nondecreasing functions of r.

2. $D(\rho, \rho', r)$ is less than for $\rho(r) = r$ for all r, i.e. $D(\rho, \rho', r) < \sqrt{1 + r^2}$

3. $(\rho'/\rho)r\sqrt{1 + r^2} > 1$

4. The function ρ' is concave.

The first assumption is motivated by the desire to avoid larger distortions closer to the center of the picture, i.e. to preserve the same type of distribution of distortions as for the limiting cases: $\rho(r) = r$ and $\rho(r) = DP(r)$.

The second assumption follows from inequalities (3.3.5).

The third assumption means that there is no "overcompression" in the radial direction. Consider an image of a small sphere. For perspective projection it is an ellipse with its major axis oriented along the direction to the center of the picture. For $DP(r)$ the image is a circle. We would expect intermediate projections to produce images of intermediate shapes, i.e. ellipses with major axis still oriented along the direction to the center but with smaller aspect ratio.

The last assumption is motivated by the fact that in a regular perspective image the amount of the shape distortion increases towards the edges, and the compression $(1/\rho')$ should increase.

$K(\rho, \rho', \rho'', r) = 3K_1^2 + \frac{\rho''^2}{\rho'^4}$, where

$$K_1 = \frac{\frac{1}{r}\left(\frac{\rho}{r} - \rho'\right)}{\rho'^2}$$

Due to Assumption 1, K_1 is positive. Note that $\rho''/\rho'^2 = -(1/\rho')'$. As $\rho'' \leq 0$ (Assumption 4), $|\rho''|/\rho'^2 = (1/\rho')'$.

Let $u = \frac{1}{\rho'}$. Then

$$K_1 = \frac{\rho u^2}{r^2} - \frac{u}{r}; \quad K = 3K_1^2 + (u')^2$$

We have assumed that K increases. Suppose $u' \geq 0$ decreases at some point; than K_1 should increase. Therefore, at that point $\frac{dK_1}{dr} > 0$. Differentiating, we get the following expression for the derivative:

$$\frac{dK_1}{dr} = \frac{1}{r}\left(\frac{2}{r}\left(u - \frac{\rho}{r}u^2\right) + u'\left(2\frac{\rho}{r}u - 1\right)\right)$$

It is non-negative if

$$u' \geq \frac{2u}{r}\frac{\frac{\rho}{r}u - 1}{2\frac{\rho}{r}u - 1}$$

Suppose that u' increases on $]r_0, 1]$. Then

$$u' = \max_{[r_0,1]} u' \geq \frac{1}{1-r_0}\int_{r_0}^{1} u' du = \frac{(u(1) - u(r_0))}{(1-r_0)} \tag{4.14}$$

If we choose $[r_0, 1]$ to be the maximal interval on which u' doesn't decrease, it will decrease on an interval $[r_0 - \epsilon, r_0]$. As $u'(r)$ increases on $[r_0, 1]$, the following estimate can be made:

$$u'(1) \geq u'(r_0) \geq \max\left(\frac{u(1) - u(r_0)}{(1-r_0)}, \frac{2u(r_0)}{r_0}\frac{\frac{\rho(r_0)}{r_0}u(r_0) - 1}{2\frac{\rho(r_0)}{r_0}u(r_0) - 1}\right) \tag{4.15}$$

It remains to estimate $u(r) = 1/\rho'(r)$. First, we estimate ρ. Denote $d = \sqrt{D+1}$. Then

$$1 \leq \frac{\rho'}{\rho} r\sqrt{1+r^2} \leq d \tag{4.16}$$

Dividing by $r\sqrt{1+r^2}$ and integrating from r to 1, we get

$$\int_r^1 \frac{dr}{\sqrt{1+r^2}} \leq -\ln\rho \leq d\int_r^1 \frac{dr}{\sqrt{1+r^2}} \tag{4.17}$$

The integral in (4.17) is equal to the function $DP(r)$ defined in Section 4.3.2. Finally, we get

$$DP(r)^d \leq \rho \leq DP(r) \tag{4.18}$$

From (4.16) and (4.18) we obtain

$$\frac{r\sqrt{1+r^2}}{DPd} \leq u = \frac{1}{\rho'} \leq \frac{r\sqrt{1+r^2}}{DP^d} \tag{4.19}$$

These bounds can be directly used in (4.15), because the first bound is an increasing function of

$u(r_0)$, and the second bound is a decreasing function of $u(r_0)$. Due to the Assumption 2, the value of $u(1)$ is known: as $D(\rho', \rho, r)$ is nondecreasing, $D(\rho'(1), 1, 1) = d$, therefore, $\rho'(1) = d/\sqrt{2}$.

We can further improve the lower bound for $\mathcal{D}$ close to 0 in the following way: the estimate (4.14) still holds for maximum of u' on $[r_0, 1]$. Suppose the estimate is maximal for some r_0. Then the value of $\mathcal{K}$ will be greater than the lower estimate for K_1 on the interval $[r_0, 1]$ plus the maximum of the estimate of u'^2.

The plot of the resulting bound is shown in Figure 4.4

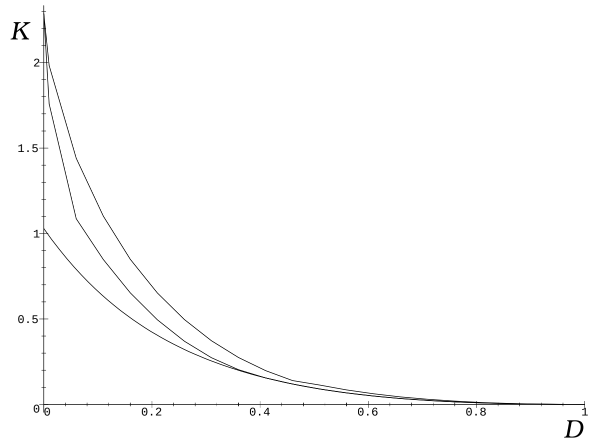

Figure 4.4: Bounds from Section 4.3.4 (upper 2 curves) and Section 4.3.3 (lower curve.) □

We can further narrow down the range of the functions that we are considering, and assume that u' increases. In this case the estimate can be further improved, as the value of $u'(1)$ will be greater than the maximum of (4.14) (Figure 4.4.)

4.3.5 Interpolations

We couldn't find a direct solution of the optimization problem (4.4), but given the estimates of the previous sections, we can evaluate interpolations between the solutions for the cases $\lambda = 1$ and $\lambda = 0$, described in Section 4.3.2.

We suggest two possible interpolations:

– *linear interpolation* described by the formula

$$\rho(r) = \lambda r/R + (1-\lambda)\frac{R(\sqrt{r^2+1}-1)}{r(\sqrt{R^2+1}-1)} \tag{4.20}$$

– *geometric interpolation* described by

$$\rho(r) = R\frac{\tan\frac{\arctan r}{2-\lambda}}{\tan\frac{\arctan R}{2-\lambda}} \tag{4.21}$$

The second case corresponds to the following geometric construction: $DP(r)$ is the projection of the plane to the unit sphere from the center of the sphere, followed by reprojection from the pole. If we move the point from which we reproject from the sphere to the plane from the pole to the center we get a smooth interpolation from r to $DP(r)$ described by (4.21).

If we compare the K–D curves for these interpolations with the estimates derived in the previous sections (both families of functions satisfy the assumptions for all estimates), we can see that not much improvement can be gained by finding better approximations to the solution of the optimization problem (Figure 4.5.)

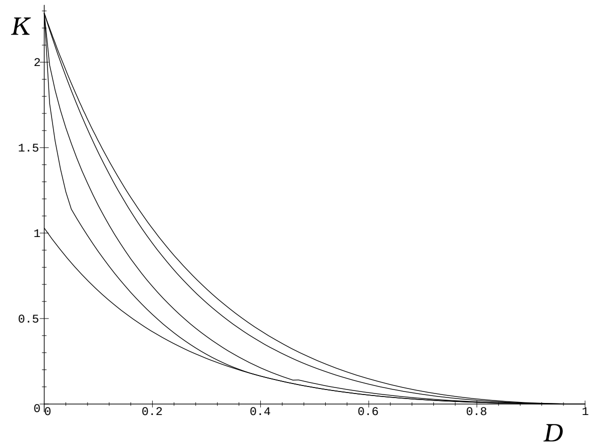

Figure 4.5: The K-D curves from top to bottom are linear interpolation, geometric interpolation, estimate from Section 4.3.3 estimate from Section 4.3.4 (lower curve.) □

It is important to remember that perception of curvature and of some shapes (such as spheres and rectangular corners) has a threshold nature. Therefore, once a perceptual requirement is violated by a sufficiently large margin, "perceptual saturation" is likely to occur and even large differences do not make much difference. Our first estimate was made without any assumptions about ρ, and it still indicates that for low values of $\mathcal{D}$, $\mathcal{K}$ should be quite large.

Nevertheless, the difference between the interpolations that we propose, and the estimates is still considerable, and it is at least of theoretical interest to obtain better estimates of the optimal (K, D) pairs and of the optimal solution.

4.3.6 Extentions

Both one-parameter families that we have described can be easily extended to larger families parametrized by the function $\lambda(\phi)$. By allowing λ depending on the angle, we obtain greater flexibility in the choice of transformation appropriate for a particular picture.

The equations for the projections take the form

$$\rho(r) = \lambda(\phi)r/R + (1 - \lambda(\phi))\frac{R(\sqrt{r^2+1} - 1)}{r(\sqrt{R^2+1} - 1)} \tag{4.22}$$

$$\rho(r) = R\frac{\tan\frac{\arctan r}{2-\lambda(\phi)}}{\tan\frac{\arctan R}{2-\lambda(\phi)}} \tag{4.23}$$

In this case, $\lambda(\phi)$ is similar to the user-specified weights described in Section 3.3.5. If we assume that λ changes slowly, the expressions for direct view and curvature error functions can be applied and for each direction θ the problem can be solved independently. Therefore, optimal values for each direction are close to those obtained for the axially-symmetric case.

For fast changes in λ, however, this does not apply, and two-dimensional optimization is neccessary to compare the family to the optimal solution as defined in Section 3.3.5.