Importance and Discrete Three Point Transport

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Abstract

Global illumination within nondiffuse environments is ideally suited for computation under importance and radiance driven hierarchical refinement: a transport interaction is significant to the view from the eye only if it lies within paths of directional reflection of both radiance originating at a light source, and importance originating at the eye.

We have recently developed a hierarchical algorithm for global illumination based on a radiance formulation for three point transport, and have implemented this algorithm over environments containing surfaces exhibiting glossy reflection. In this paper we derive the adjoint to our original transport formulation, and describe the preliminary results of application of this adjoint in the form of an importance driven version of our glossy implementation. These results show a significant reduction in transport computation, and indicate that an importance and radiance driven hierarchical technique possesses great promise for the development of efficient global illumination algorithms for general reflection.

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Key Words: adjoint transport equation, adaptive meshing, global illumination.

1 Introduction

In recent work, we have developed a hierarchical algorithm for global illumination based on a radiance formulation for discrete three point transport, and have implemented and analyzed this algorithm over environments containing surfaces exhibiting glossy and diffuse reflection [2]. In this paper we derive the adjoint to our transport formulation, and describe the preliminary results of application of this adjoint in the form of an importance driven version of our glossy implementation. These results show a significant reduction in transport computation, and indicate that an importance and radiance driven hierarchical technique possesses great promise for the development of efficient global illumination algorithms for general reflection.

In the following section we will provide a brief overview of previous work in discrete three point transport, and the application of importance to hierarchical illumination. In Section 3 we derive the adjoint to our transport formulation, and discuss the interpretation of the resulting equation. In Section 4 we describe the application of the adjoint to our algorithm, and discuss the distribution of importance within a transport interaction hierarchy. Finally, in Section 5 we present preliminary results of importance and radiance driven computation over glossy reflection.

Finally, we note that our derivation of the adjoint and discussion of importance is specifically oriented toward discrete three point transport, and in particular to the iterative solution technique we employ for solution over this transport. A very interesting, and more general and fundamental, discussion of the adjoint formulation for transport is presented by Christensen et al. in a paper appearing in this proceedings [3].

2 A Brief Overview of Previous Work

2.1 Discrete Three Point Transport

Consider an environment consisting of k polygonal patches discretized into n sufficiently small elements — let A_j and A_k be two such elements. Then, as shown in [2], the following radiance formulation for three point transport may be obtained:

$$L_{jk} = E_{jk} + \sum_{i=1}^{n} R_{kji} L_{ij}$$

This equation states that the radiance at area A_j in the direction of area A_k is equal to the radiance emitted by A_j in the direction of A_k, plus, for every area A_i, the radiance at A_i in the direction of A_j multiplied by the area reflectance R_{kji}. Area reflectance, R_{kji}, is the fraction of the radiant flux transported from A_k incident to A_j that is reflected in the direction of area A_i. As discussed in [2], the accuracy of estimators for R_{kji} is improved as the sizes of A_i, A_j, and A_k decrease relative to their distances apart.

This formulation, combined with its improvement under subdivision, leads directly to a hierarchical refinement algorithm for three point transport. Under the algorithm, patch-patch transport *interactions* play the role analogous to patches under hierarchical radiosity [4, 5] — refinement is carried out over interaction pairs until error is satisfied, in combination with a gathering iteration over interactions for the computation of actual transport.

We have implemented the algorithm for glossy reflection under a highly simplified Torrance-Sparrow model [6, 2], estimating error by employing techniques described in [4] combined with a simple interval technique over enclosing cones [2]. We have shown theoretic growth in transport computations under the model to be $O(n + k^3)$ — this growth is exhibited in experimental trials. Naive application of three point transport would require $O(n^3)$ computation.

2.2 Importance and Hierarchical Illumination

The brief sketch provided here is based directly on the excellent discussion presented in [5]. Consider the transport equation:

$$TL = E$$

where T is a linear transport operator, E is the emission term, and L the desired final radiosities or radiances. We may write the adjoint to this equation as:

$$T^* I = N$$

in which T^* is the adjoint to T, I is importance — the fraction of the illumination at each element that ultimately reaches the eye, and N is "emitted" importance (the dual to E, emitted illumination). For a given scalar, linear view function $v(L)$, N is set such that $v(L) = N^{\mathrm{T}} L$.

Then, as discussed in [5], for a computed approximation $\tilde{T} = T + \Delta T$ to the exact transport operator, the resulting error as viewed from the eye may be estimated as:

$$\tilde{v}(L - \tilde{L}) = \tilde{I}^{\mathrm{T}} \Delta T \tilde{L}$$

where $\tilde{L}$ and $\tilde{I}$ are the illumination and importance solutions computed under $\tilde{T}$.

Smits et al. have developed an importance and radiosity driven hierarchical radiosity algorithm that solves for transport of both illumination and importance. As predicated by the above, hierarchical refinement is controlled by error computed as transport error multiplied by estimated importance and radiosity. The algorithm thus allows accurate computation of radiosity significant to the view from the eye, and avoids wasted refinement that does not influence this view.

3 Importance and the Adjoint to the Transport Equation

In the superb paper of Smits et al. [5] on importance driven radiosity, refinement and illumination are computed within a maze — within such an environment, many light interactions contribute to the view only after multiple reflections, and are thus relatively "unimportant", requiring little refinement. The preferred directionality of more specular reflection creates a

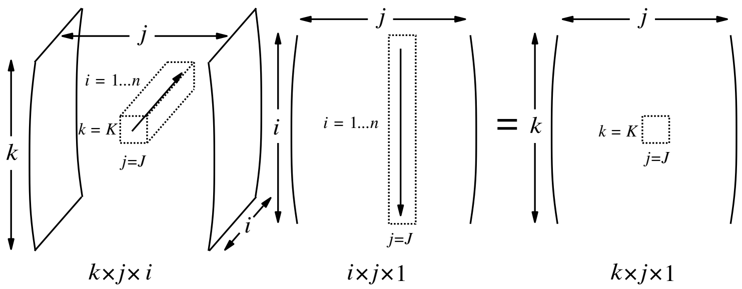

Figure 1: A 3D matrix "product": the row vector (into the page) at $j = J$ and $k = K$ in the 3D matrix multiplies the column at $j = J$ in the 2D matrix, to give the entry at $k = K, j = J$ in the result.

natural maze of interactions within transport space, in which only a limited number of paths substantially contribute to, or even lead to, the eye. Imagine importance radiating from the eye and interreflecting among surfaces in a near specular fashion: only interaction paths that lie along specular reflections will be strongly "illuminated" by importance, and thus require any degree of refinement. Under directional reflection, even within a fairly simple environment, there is a very high fraction of unimportant transport paths, much higher than exhibited under diffuse reflection. We can thus expect importance weighting to be even more significant for three point transport over glossy reflection than for diffuse reflection under radiosity.

3.1 Derivation of the Adjoint

Recall the system of equations that describe the radiance formulation for three point transport:

$$E_{jk} = L_{jk} - \sum_{i=1}^{n} R_{kji} L_{ij}$$

we would like to rewrite this system in the form:

$$TL = E$$

Note that under the given system we may decompose T into the operators $T = (I_d - R)$, where I_d is simply the identity operator, and R is a "reflectance" operator accounting for the R_{kji} terms. We will first examine the derivation of R: the three dimensional character of R_{kji} over indices i, j, k leads (a computer scientist at least) naturally to a sort of three dimensional matrix or array. One could imagine a product over an $n \times n \times n$ R_{kji} "matrix" and an $n \times n \times 1$ L_{ij} matrix producing an $n \times n \times 1$ matrix of reflected radiances, such as shown in figure 1.

How one might manipulate such a matrix, however, is not be immediately clear. Furthermore, it is not possible to incorporate the identity operator into the shown "product" —

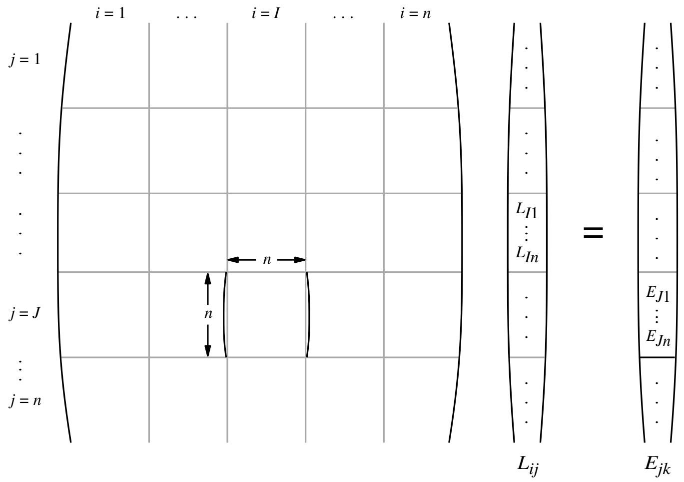

Figure 2

each entry R_{kji} in the matrix multiplies some L_{ij} to yield result E_{jk}: such multiplication may not yield identity unless $i = j = k$.

These problems do not arise if the operator R is rewritten as a standard two-dimensional matrix. The $O(n^3)$ entries in R_{kji} may be placed within an $n^2 \times n^2$ matrix, the remaining entries set to zero. Given an $n^2 \times n^2$ transport matrix, we employ an $n^2 \times 1$ column vector for L, the resulting matrix-vector product yielding a similar $n^2 \times 1$ column vector for E. We choose to order the entries in L and E with the first index as most significant, that is, $(L_{11}, L_{12}, \ldots, L_{1n}, L_{21}, L_{22}, \ldots\ldots, L_{nn})$.

Consider the entries in the $n^2 \times n^2$ matrix representation for R (figure 2). For a given $i = I$, the consecutive L_{Ij} entries in the L_{ij} vector determine a column of width n within the reflectance matrix. Similarly, for a given $j = J$, the consecutive E_{Jk} entries in E_{jk} determine a reflectance row, also of width n. Indices i and j together partition the matrix into n^2 $n \times n$ submatrices.

A given J-I submatrix represents the operator's action upon the element L_{IJ} in L_{ij}. Note that, within the submatrix, only elements in the J^{th} column multiply L_{IJ}. This column should thus contain $R_{kJI} = \{R_{1JI}, R_{2JI}, \ldots, R_{nJI}\}$, the multipliers of L_{IJ} within the system of equations (figure 3). A complete reflectance matrix for the system under $n = 3$ is given in figure 4. Given a two dimensional representation for R, the identity operator immediately follows, and we thus have the desired $T = I_d - R$.

T is a real matrix, and thus the adjoint to T is simply its transpose, giving the adjoint equation:

$$T^{\mathrm{T}} I = (I_d - R^{\mathrm{T}}) I = N$$

$$
\begin{array}{cc}
 & I \\
 & \begin{array}{cccccccc} 1 & 2 & \dots & & J & & \dots & n \end{array} \\
J & \begin{pmatrix}
0 & 0 & \dots & 0 & R_{1JI} & 0 & \dots & 0 \\
0 & 0 & \dots & 0 & R_{2JI} & 0 & \dots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 0 & R_{nJI} & 0 & \dots & 0
\end{pmatrix}
\end{array}
$$

Figure 3

$$
\begin{pmatrix}
\begin{pmatrix} R_{111} & 0 & 0 \\ R_{211} & 0 & 0 \\ R_{311} & 0 & 0 \end{pmatrix} & \begin{pmatrix} R_{112} & 0 & 0 \\ R_{212} & 0 & 0 \\ R_{312} & 0 & 0 \end{pmatrix} & \begin{pmatrix} R_{113} & 0 & 0 \\ R_{213} & 0 & 0 \\ R_{313} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & R_{121} & 0 \\ 0 & R_{221} & 0 \\ 0 & R_{321} & 0 \end{pmatrix} & \begin{pmatrix} 0 & R_{122} & 0 \\ 0 & R_{222} & 0 \\ 0 & R_{322} & 0 \end{pmatrix} & \begin{pmatrix} 0 & R_{123} & 0 \\ 0 & R_{223} & 0 \\ 0 & R_{323} & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & R_{131} \\ 0 & 0 & R_{231} \\ 0 & 0 & R_{331} \end{pmatrix} & \begin{pmatrix} 0 & 0 & R_{132} \\ 0 & 0 & R_{232} \\ 0 & 0 & R_{332} \end{pmatrix} & \begin{pmatrix} 0 & 0 & R_{133} \\ 0 & 0 & R_{233} \\ 0 & 0 & R_{333} \end{pmatrix}
\end{pmatrix}
$$

Figure 4

Consider the action of this operator to produce a given N_{JK}. Transposition of R induces a transposition of its submatrices; N_{JK} is thus determined by the transpose of the submatrix at $j = K$ and $i = J$ in the original R. By the construction of this submatrix, we have:

$$N_{JK} = I_{JK} - R_{1KJ}I_{K1} - R_{2KJ}I_{K2} - \cdots - R_{nKJ}I_{Kn}$$

this gives us the equation for transport of importance:

$$I_{jk} = N_{jk} + \sum_{i=1}^{n} R_{ikj}I_{ki}$$

3.2 Discussion

Recall the equation for three point transport of radiance:

$$L_{jk} = E_{jk} + \sum_{i=1}^{n} R_{kji}L_{ij}$$

This equation is relatively straightforward: the radiance at A_j in the direction of A_k depends on the radiance A_j receives from all other areas A_i, multiplied by a function of reflectance between the three areas. This reflectance has been "reversed", as with the form factor under radiosity, through the use of a reciprocity relation to eliminate area dependence.

In contrast, an intuitive understanding of the adjoint equation may not be immediately obvious: consider three areas A_I, A_J, and A_k; we shall assume that I_{IJ} is of great magnitude,

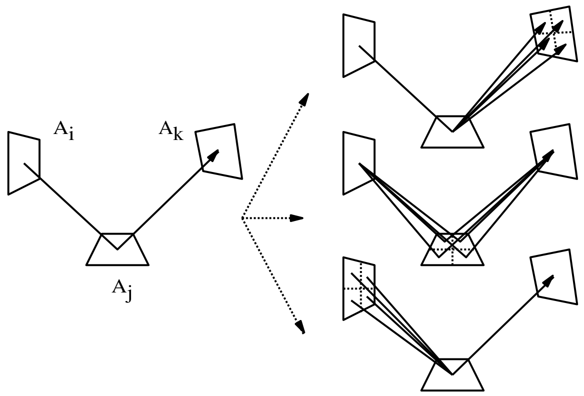

Figure 5

that there is no importance between any other pairing of areas, and that $N_l = 0$ for $l = I, J, K$. Then, by the adjoint equation, we have: $I_{JK} = I_{KJ} = I_{IK} = 0$, but

$$I_{KI} = N_{KI} + R_{JIK} I_{IJ} = R_{JIK} I_{IJ} > 0$$

assuming R_{JIK} is of nonzero value.

This result may seem counterintuitive, as one might expect importance I_{IJ} to be reflected from A_J toward A_K, and thus contribute to I_{JK} rather than I_{KI}. What should be noted is that I_{IJ} may be understood as the importance that *arrives* at area A_I *from* area A_J, in contrast to L_{IJ}, the radiance at A_I outgoing in the direction of A_J. Importance arriving at area A_I is thus reflected at A_I under R_{JIK} toward A_K, contributing to I_{KI}. Radiance and importance are represented and transported in opposing directions; both, however, are reflected under three point transport as governed by area reflectance.

4 Application

Given the adjoint equation, we have developed an importance driven version of our hierarchical algorithm for glossy reflection under three point transport [2], directly analogous to the importance driven radiosity algorithm presented in [5]. Refinement over interaction pairs (and corresponding area triples) is governed by transport error weighted by estimated transport radiance and importance. If this error exceeds user specified bounds, one or both interactions are subdivided and refinement is recursively applied (figure 5); otherwise the interactions are *linked*. Actual transport is computed via dual gather/shoot iterations, in which each interaction gathers radiance from, and gives importance to, those interactions with which it has been linked.

A gather/shoot iteration results in received radiance and importance scattered throughout each interaction hierarchy. These quantities must be distributed and accounted for over all ancestors and descendents of each receiving interaction, in order to correctly maintain the hierarchical representation of the dual systems. We employ a distribution algorithm similar to those presented in [4, 5] for radiosity and importance over patch/element hierarchies: received radiance and importance is "pushed" to the leaf interactions within each hierarchy to ensure propagation to all descendents, and then "pulled" and distributed back up from the leaves through all higher level interactions to their common ancestor at the root. To correctly implement this algorithm, we must understand how the importance at a given interaction is related to the importance at its parent and at its children.

4.1 Importance within a Hierarchy

As is shown in [1], radiance may be pushed unchanged from parents to children within an interaction hierarchy, and area averaged as it is pulled from child to parent. In this section we examine the distribution of importance within interaction hierarchies.

Consider a transport interaction from patch A_j to patch A_k. Under the adjoint equation the importance at A_j in the direction of A_k is determined by area reflectance R_{ikj}. The dependence of R_{ikj} upon subdivision of A_j and A_k thus determines the distribution of received importance within the interaction hierarchy.

For a given bidirectional reflectance-distribution function f_r, area reflectance R_{ikj} is defined such that [2]:

$$\pi A_k F_{ki} R_{ikj} = \int_{A_i}\int_{A_k}\int_{A_j} f_r(x,x',x'')G(x,x')G(x',x'')dx''dx'dx$$

where F_{ki} is the diffuse form factor from A_k to A_i, and G is the related quantity:

$$G(x,x') = \frac{\cos\theta_i \cos\theta_k}{|x-x'|^2}$$

We first consider subdivision of A_k, yielding children $A_{k_1}, A_{k_2}, \ldots, A_{k_c}$. We may rewrite the above expression for R_{ikj} as:

$$\begin{aligned}
\pi A_k F_{ki} R_{ikj} &= \int_{A_i}\int_{A_k}\int_{A_j} f_r(x,x',x'')G(x,x')G(x',x'')dx''dx'dx \\
&= \int_{A_i}\int_{A_{k_1}}\int_{A_j} f_r(x,x',x'')G(x,x')G(x',x'')dx''dx'dx \\
&\quad + \cdots + \int_{A_i}\int_{A_{k_c}}\int_{A_j} f_r(x,x',x'')G(x,x')G(x',x'')dx''dx'dx \\
&= \pi A_{k_1} F_{k_1 i} R_{ik_1 j} + \cdots + \pi A_{k_c} F_{k_c i} R_{ik_c j}
\end{aligned}$$

Under the subdivision criteria employed in [2], we may assume that if the interaction from A_j to A_k is employed in transport, then areas A_j and A_k are well separated, and thus

$F_{ki} \approx F_{k_d i}$, for $d = 1 \ldots c$ [5]. We then divide through by $\pi A_k F_{ki}$, yielding:

$$R_{ikj} = \frac{A_{k_1}}{A_k} R_{ik_1 j} + \cdots + \frac{A_{k_c}}{A_k} R_{ik_c j}$$

Thus, given distribution by area reflectance, the importance at a parent interaction is simply the area average of the importance at its children, under subdivision of A_k.

In contrast, subdivision of A_j induces a simple sum rather than an average:

$$\begin{aligned}
\pi A_k F_{ki} R_{ikj} &= \int_{A_i}\int_{A_k}\int_{A_j} f_r(x, x', x'') G(x, x') G(x', x'') dx'' dx' dx \\
&= \int_{A_i}\int_{A_k}\int_{A_{j_1}} f_r(x, x', x'') G(x, x') G(x', x'') dx'' dx' dx \\
&\quad + \cdots + \int_{A_i}\int_{A_k}\int_{A_{j_c}} f_r(x, x', x'') G(x, x') G(x', x'') dx'' dx' dx \\
&= \pi A_k F_{ki} R_{ikj_1} + \cdots + \pi A_k F_{ki} R_{ikj_c}
\end{aligned}$$

dividing through by $\pi A_k F_{ki}$ we have:

$$R_{ikj} = R_{ikj_1} + \cdots + R_{ikj_c}$$

In a similar fashion, it may be shown that importance received at a given parent interaction may be pushed unchanged to its children under subdivision of A_k, but should be distributed by proportional area given subdivision of A_j.

5 Results

The image in figure 6 shows preliminary results of application under radiance and importance weighting. The environment contains four glossy reflectors: the broad face of each of the three "slabs", and the top of the central cube. In addition to the reflections seen within the slabs, note the light originating from the lamp at left, reflected from the cube, and over the upper area of the wall at right. There are potentially 222 billion discrete transport triples within this environment at maximum allowed refinement — the application employs 70,995, a reduction to 3 hundred-thousandths of 1 percent. The image was computed after seven complete iterations (gathers/shoots to all interactions), and total time just over three minutes on a Silicon Graphics Indigo workstation with a 50MHz R4000 processor.

Figure 6				
Max Elements		15138		
Max Triples		222385209344		
Computed	subpatches	elements	triples	time
glossy exponent = 500	6479	4866	70995 (0.00003%)	3m13s

Figure 6

6 Acknowledgements

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