On the Form Factor between Two Polygons

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Abstract

Form factors are used in radiosity to describe the fraction of diffusely reflected light leaving one surface and arriving at another. They are a fundamental geometric property used for computation. Many special configurations admit closed form solutions. However, the important case of the form factor between two polygons in three space has had no known closed form solution. We give such a solution for the case of general (planar, convex or concave, possibly containing holes) polygons.

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Additional Key Words and Phrases: Closed form solution; form factor; polygons.

1 Introduction

When using the radiosity technique to create images the form factor plays a central role. It describes the fraction of radiation diffusely emitted from one surface reaching another surface. The accurate computation of form factors is the central theme in many recent papers. Goral et al. [4], who introduced radiosity to the computer graphics community, used numerical contour integration to compute form factors between polygons. Cohen and Greenberg [3] took visibility into account with their hemicube algorithm. More recent hierarchical and adaptive algorithms compute still more accurate form factors [10; 5]. Nishita and Nakamae [8] and Baum et al. [2] have used an exact solution for the form factor between a differential surface element and a polygon. Most radiosity algorithms are restricted to polygonal environments, and so a closed form solution for the form factor between polygons is potentially of great utility.

The history of computing form factors is very long. A closed form expression for the form factor between a differential surface element and a polygon was found by Lambert in 1760 [7]. Lambert proceeded to derive the form factor for a number of special configurations among them the form factor between two perpendicular rectangles sharing a common edge. He writes about the latter derivation:

> Although this task appears very simple its solution is considerably more knotted than one would expect. For it would be very easy to write down the differential expression of fourth order, which one would need to integrate four fold; but the highly laborious computation would fill even the most patient with disgust and drive them away from the task.

Other workers have derived closed form solutions for the form factors between many different geometric configurations and these can be found in standard textbooks. However, we are not aware of a closed form solution for the form factor between two general polygons. Thus, this problem has remained open for over 230 years.

In this paper we present a formula for the form factor integral between two general polygons. The derivation of this formula is quite involved, and the interested reader is referred to [9] for a detailed derivation. The purpose of this paper is to bring this result to the attention of the graphics community.

2 Closed form solution

The form factor integral can be reduced to a double contour integral by two applications of Stokes' theorem [6]

$$\begin{aligned}\pi A_1 F_{12} &= \int_{A_1}\int_{A_2} \frac{\cos\theta_1 \cos\theta_2}{\|\vec{r}\|^2}\, dA_2\, dA_1 \\ &= \tfrac{1}{4}\int_{\partial A_1}\int_{\partial A_2} \ln(\vec{r}\cdot\vec{r})\, d\vec{x}_2 \cdot d\vec{x}_1\end{aligned}$$

where θ_1, θ_2 are the angles between the normal vector of the respective surface and a radius vector $\vec{r}$, which connects two points on the surfaces. The above equation holds for all surfaces such that every point on either surface sees the same contour of the other surface.

In the case of polygons P_1 and P_2 the contour integral reduces to a sum of double line integrals over all pairwise combinations of edges

$$4\pi A_{P_1} F_{P_1 P_2} = \sum_{E_i\, E_j} \cos\angle E_i E_j \int_{E_i}\int_{E_j} \ln(\vec{r}\cdot\vec{r})\, ds_j\, dt_i$$

Ignoring the factor $\cos\angle E_i E_j$ we are left with the task of giving a solution to integrals of the general form $\int_0^{c_2}\int_0^{c_0} \ln f(s,t)\, ds\, dt$. c_0 and c_2 are the lengths of the edges over which a given double contour integral is taken and $f(s,t) = s^2 + c_1 st + t^2 + c_3 s + c_4 t + c_5$ is the bi-quadratic form which arises from the expansion of the dot product (see Table 2 for definitions of all variables). If the two line segments lie in a common plane we can factor $f(s,t)$ into two bi-linear forms and a solution is readily obtained with standard integration tables (see [9]). Lines in general position lead to the following result:

$$\begin{aligned}&\int_0^{c_2}\int_0^{c_0} \ln f(s,t)\, ds\, dt \\ &= \left\{\left[(s+\frac{c_3}{2})\mathrm{G}(f(.,t))(s) + \frac{c_1}{2}\mathrm{H}(f(.,t))(s)\right]\Big|_{s=0}^{s=c_0}\right\}\Bigg|_{t=0}^{t=c_2} \\ &\quad -2c_0c_2 + c_{14}c_{15}\left\{\left[\pi(2k(s)+1)\mathrm{M}(t)\right.\right. \\ &\quad -i\left(\mathrm{L}(-c_{17}(s))(t) + \mathrm{L}(-c_{18}(s))(t)\right. \\ &\quad \left.\left.\left.-\mathrm{L}(c_{17}(s))(t) - \mathrm{L}(c_{18}(s))(t)\right)\right]\Big|_{s=0}^{s=c_0}\right\}\Bigg|_{t=\sqrt{\frac{c_{13}}{\bar{c}_{13}}}}^{t=\sqrt{\frac{c_{13}+c_2}{\bar{c}_{13}+c_2}}}\end{aligned}$$

where $k(s) \in \{-1, 0, 1\}$ according to the particular branchcut of the complex logarithm choosen in L. The auxiliary functions G, H, L, and M are given in Table 1.

3 An example

We have implemented our closed form solution in Mathematica [11] (this code is available from **ps@princeton.edu**). The

$$
\begin{aligned}
\mathrm{L}(b)(y) &:= \int^y t^2(1-t^2)^{-3}\ln(b+t)\,dt &&= \tfrac{1}{16}\left[\tfrac{-b\ln(y-1)}{(b+1)^2} - \tfrac{b\ln(1+y)}{(b-1)^2} + \left(\tfrac{2(b+y)(1+by)[(b-y)^2+(by-1)^2]}{(b^2-1)^2(y^2-1)^2} + \ln\tfrac{(1-y)(1-b)}{(1+y)(1+b)}\right)\ln(b+y)\right.\\
&&&\quad \left. + \tfrac{2(b-y)}{(b^2-1)(y^2-1)} + \mathrm{Li}_2\left(\tfrac{1-y}{1+b}\right) - \mathrm{Li}_2\left(\tfrac{1+y}{1-b}\right)\right]\\
\mathrm{M}(y) &:= \int^y t^2(1-t^2)^{-3}\,dt &&= \tfrac{1}{16}\left[4y(y^2-1)^{-2} + 2y(y^2-1)^{-1} + \ln\tfrac{y-1}{y+1}\right]\\
\mathrm{G}(q)(y) &:= \int^y \ln q(t)\,dt &&= \tfrac{q'(y)}{2a}\ln q(y) - 2y + \tfrac{d}{a}\tan^{-1}\tfrac{q'(y)}{d}\\
\mathrm{H}(q)(y) &:= \int^y t\ln q(t)\,dt &&= \left(\tfrac{y^2}{2} + \tfrac{c}{2a} - \tfrac{b^2}{4a^2}\right)\ln q(y) - \tfrac{y(ay-b)}{2a} - \tfrac{bd}{2a^2}\tan^{-1}\tfrac{q'(y)}{d}
\end{aligned}
$$

Table 1: Four auxiliary integrals needed in the solution. Notice that $\mathrm{L}(b)(y)$ uses the dilogarithm [1], $\mathrm{Li}_2(z) = \sum_1^\infty \frac{z^k}{k^2}$, $\frac{d}{dz}\mathrm{Li}_2(z) = -\frac{\ln(1-z)}{z}$. In G and H the argument q is an arbitrary quadratic polynomial $q(t) = at^2 + bt + c$ and $d = \sqrt{4ac - b^2}$.

$$
\begin{aligned}
c_0 &= \|E_j\| & c_{13} &= \tfrac{c_{11} - \sqrt{c_{11}^2 - 4c_{10}c_{12}}}{2c_{10}}\\
c_1 &= -2\vec{d}_i\cdot\vec{d}_j & c_{14} &= \tfrac{\sqrt{c_{11}^2 - 4c_{10}c_{12}}}{c_{10}}\\
c_2 &= \|E_i\| & c_{15} &= \sqrt{c_{10}}c_{14}\\
c_3 &= -2\vec{d}_j\cdot(\vec{p}_i - \vec{p}_j) & c_{16}(s) &= c_1c_{13} - c_3 - 2s\\
c_4 &= 2\vec{d}_i\cdot(\vec{p}_i - \vec{p}_j) & c_{17}(s) &= \tfrac{-c_{15} + \sqrt{c_{15}^2 - 4|c_{16}(s)|^2}}{2i\bar{c}_{16}(s)}\\
c_5 &= \|\vec{p}_i - \vec{p}_j\|^2 & c_{18}(s) &= \tfrac{-c_{15} - \sqrt{c_{15}^2 - 4|c_{16}(s)|^2}}{2i\bar{c}_{16}(s)}\\
c_{10} &= 4 - c_1^2\\
c_{11} &= 4c_4 - 2c_1c_3\\
c_{12} &= 4c_5 - c_3^2
\end{aligned}
$$

Table 2: All expressions for two edges E_{ij} with parameterization $\vec{x}_i(t) = \vec{p}_i + t\vec{d}_i$ and $\vec{x}_j(s) = \vec{p}_j + s\vec{d}_j$ ($\|\vec{d}_{i,j}\| = 1$).

implementation requires some care because of the complexities of the functions that are involved.

A simple example, which requires the full power of our formula, concerns the form factor between two equal width rectangles sharing an edge with an enclosing angle $\theta \in [0, \pi]$. The configuration is illustrated in Figure 1 together with the form factor as a function of θ for different aspect ratios $l = \frac{a}{b}$ (common edge length b).

4 Conclusion

We have given a closed form solution for the form factor between two general polygons. This solution is non-elementary since it involves the dilogarithm function. The principal value of our solution is in determining exact answers for general polygonal configurations. This can be used in practice for reference solutions to check more efficient approximations. Baum et al. [2] have also shown that the error in the computed solution can be reduced significantly when using a closed form solution near singularities of the integrand.

There has been a long history of computing closed form expressions for form factors starting with Lambert in 1760. The literature lists many special cases for which closed form solutions exist, but hitherto no solution had been given for general polygonal configurations. The present paper closes this gap.

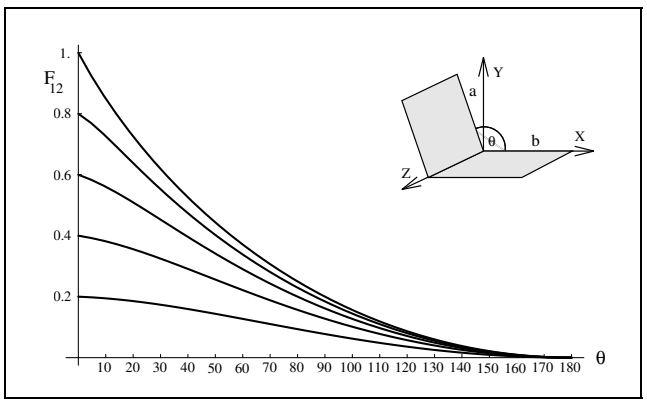

Figure 1: Geometry for two rectangles sharing a common edge with an enclosing angle of θ. The graphs show the form factor as a function of θ for edge ratios $l = \frac{a}{b}$ of .2, .4, .6, .8, and 1.0.

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