Appendix A

Proof of Theorem 5.1

Theorem 5.1

Given the integral:

$$I = \iiint h\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) dxdydz$$

where:

$$z = f(x, y)$$

and the function h is of the form:

$$h\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = e(x, y, z)\left[\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) \cdot \mathbf{v}(x, y, z)\right]$$

the function z that minimizes the integral satisfies the relation:

$$\mathbf{v} \cdot \nabla e + e\nabla \cdot \mathbf{v} = 0$$

Proof:

The solution lies in applying the calculus of variations, which has the fundamental result known as the Euler-Lagrange equation:

$$\frac{\partial h}{\partial z} - \frac{\partial}{\partial x}\frac{\partial h}{\partial(\partial z/\partial x)} - \frac{\partial}{\partial y}\frac{\partial h}{\partial(\partial z/\partial y)} = 0 \tag{A.1}$$

We begin with the first term, $\partial h/\partial z$. We can rewrite h as:

$$\begin{aligned} h &= e\left[\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)\cdot \mathbf{v}\right] \\ &= e\left[-v_x\frac{\partial z}{\partial x} - v_y\frac{\partial z}{\partial y} + v_z\right] \end{aligned}$$

where we have dropped the explicit dependencies on (x, y, z) and where the components of $\mathbf{v}$ are (v_x, v_y, v_z). Then we can compute:

$$\frac{\partial h}{\partial z} = \frac{\partial e}{\partial z}\left[-v_x\frac{\partial z}{\partial x} - v_y\frac{\partial z}{\partial y} + v_z\right] + e\left[-\frac{\partial v_x}{\partial z}\frac{\partial z}{\partial x} - \frac{\partial v_y}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial v_z}{\partial z}\right] \tag{A.2}$$

Next, we compute the intermediate partials:

$$\frac{\partial h}{\partial(\partial z/\partial x)} = -ev_x$$

and:

$$\frac{\partial h}{\partial(\partial z/\partial y)} = -ev_y$$

The remaining partials are then:

$$\begin{aligned} \frac{\partial}{\partial x}\frac{\partial h}{\partial(\partial z/\partial x)} &= -\frac{\partial(ev_x)}{\partial x} - \frac{\partial z}{\partial x}\frac{\partial(ev_x)}{\partial z} \\ &= -v_x\frac{\partial e}{\partial x} - e\frac{\partial v_x}{\partial x} - \frac{\partial z}{\partial x}\left[v_x\frac{\partial e}{\partial z} + e\frac{\partial v_x}{\partial z}\right] \end{aligned} \tag{A.3}$$

and:

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{\partial h}{\partial(\partial z/\partial y)} &= -\frac{\partial(ev_y)}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial(ev_y)}{\partial z} \\
&= -v_y\frac{\partial e}{\partial y} - e\frac{\partial v_y}{\partial y} - \frac{\partial z}{\partial y}\left[v_y\frac{\partial e}{\partial z} + e\frac{\partial v_y}{\partial z}\right] \qquad \text{(A.4)}
\end{aligned}$$

Substituting Equations A.2, A.3, and A.4 into the Euler-Lagrange equation (Equation A.1), we arrive at:

$$v_z\frac{\partial e}{\partial z} + e\frac{\partial v_z}{\partial z} + v_x\frac{\partial e}{\partial x} + e\frac{\partial v_x}{\partial x} + v_y\frac{\partial e}{\partial y} + e\frac{\partial v_y}{\partial y} = 0$$

Rearranging:

$$\left(v_x\frac{\partial e}{\partial x} + v_y\frac{\partial e}{\partial y} + v_z\frac{\partial e}{\partial z}\right) + \left(e\frac{\partial v_x}{\partial x} + e\frac{\partial v_y}{\partial y} + e\frac{\partial v_z}{\partial z}\right) = 0$$

In “del” notation, this becomes:

$$\mathbf{v}\cdot\nabla e + e\nabla\cdot\mathbf{v} = 0$$

□