Reminder: the differential of a function.

- The tangent space of $\mathbb{R}^n$ at $p$, denoted $T_p\mathbb{R}^n$. Tangent vectors of curves.
- The differential of $f : \mathbb{R}^n \to \mathbb{R}^m$ at $p$ is the matrix $Df_p \in \mathbb{R}^{m \times n}$ with components $\frac{\partial f^j}{\partial x^i}$.
- Interpretation as a linear mapping $Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$. Image of curves and their tangent vectors. Let $c : I \to \mathbb{R}^n$ be a curve with $c(0) = p$ and $\dot{c}(0) = X_p$. Then
  \[ \frac{d}{dt}f(c(t)) \bigg|_{t=0} = \left( \ldots, \sum_i \frac{\partial f^j}{\partial x^i} \circ c(t) \frac{dc^i(t)}{dt} \bigg|_{t=0}, \ldots \right) = Df_p \cdot X_p \]
- The rank of $Df_p$. Injectivity and surjectivity.
- Qualitative picture of a map of locally constant rank. Let $f : \mathbb{R}^n \to \mathbb{R}^m$.
  - If $Df_p$ is injective for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $n \leq m$ and we can “modify” $f$ as follows: there exist smooth bijections with smooth inverses (a.k.a. diffeomorphisms) $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi : \mathbb{R}^m \to \mathbb{R}^m$ (actually defined on suitable open sets of $\Omega$ and $f(\Omega)$) so that the map $\tilde{f} := \psi \circ f \circ \phi^{-1}$ has the form
    \[ \tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0) \]
    for all $x := (x^1, \ldots, x^n)$ in the domain of $\phi$.
  - If $Df_p$ is surjective for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $n \geq m$ and a similar modification of $f$ has the form
    \[ \tilde{f}(x^1, \ldots, x^n, x^{m+1}, \ldots, x^m) = (x^1, \ldots, x^m) \]
    for all $x := (x^1, \ldots, x^n)$ in the domain of $\phi$. Note that $\tilde{f}$ can be many-to-one since, for instance, we have $\tilde{f}^{-1}(0) = \left\{(0, \ldots, 0, x^{m+1}, \ldots, x^n) : x^i \in \mathbb{R} \text{ for each } i \right\}$.
  - If $Df_p$ is bijective for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $n = m$ and a similar modification of $f$ has the form
    \[ \tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^n) \]
    for all $x := (x^1, \ldots, x^n)$ in the domain of $\phi$. Note that $\tilde{f}$ and thus $f$ are locally bijective.
  - If $Df_p$ has rank $k$ for all $p \in \Omega \subseteq \mathbb{R}^n$ then we must have $k \leq \min(n, m)$ and a similar modification of $f$ has the form
    \[ \tilde{f}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0) \]
    for all $x := (x^1, \ldots, x^n)$ in the domain of $\phi$.

- Proofs are based on the inverse and implicit function theorems.

**InvFT.** If $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth with $Df_p$ bijective, then $f$ is invertible on a neighbourhood of $p$. Note that $Df_p$ is bijective at $p$ if and only if $\det(Df_p) \neq 0$. This is an open condition so we actually obtain a stronger result than above.

**ImpFT.** If $F : \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth with $D_1F_{(p,q)}$ bijective and $F(p,q) = 0$, then the equation $F(x,y) = 0$ can be solved for points $(x,y)$ near $(p,q)$ in the following sense. There exists a function $g : \mathbb{R}^k \to \mathbb{R}^n$ defined in a neighbourhood of $q$ giving us $y = g(x)$ for which $q = g(p)$ and also $F(x, g(x)) = 0$. Note that we can compute $Dg_x$ in terms of $D_1F_{(x,g(x))}$ and $D_2F_{(x,g(x))}$. Example: $F(x,y,z) = x^2 + y^2 + z^2 - 1$. 

Three kinds of surfaces.

- Common representations of surfaces in \( \mathbb{R}^3 \).
- Graphs of functions \( f : \mathbb{R}^2 \to \mathbb{R} \). Examples: planes, upper hemisphere.
- Level sets of functions \( F : \mathbb{R}^3 \to \mathbb{R} \). Examples: the whole sphere. Conic sections. Graphs as the zero level set of \( F(x, y, z) := z - f(x, y) \). Writing a level set as a graph — when this is possible, and the relation to ImpFT.
- Parametric surfaces \( \sigma : U \to \mathbb{R}^3 \) where \( U \subseteq \mathbb{R}^2 \) is an open domain in the plane and \( \sigma(u^1, u^2) := (\sigma^1(u^1, u^2), \sigma^2(u^1, u^2), \sigma^3(u^1, u^2)) \). Examples: sphere, torus. Graphs as parametrized surfaces \( (x, y) \mapsto (x, y, f(x, y)) \). Relation with level sets: \( F(\sigma(u)) = \text{const} \) for all \( u \in U \).

Suppose you come across a surface in \( \mathbb{R}^3 \), what representation do you choose to describe it mathematically? Each representation has its limitations.

- Not every surface is a graph.
- How do you find a level set function? Or if you know the level set function, how do you solve it? You have to solve equations! E.g. if \( F(x, y, z) = 0 \) you need to extract \( z = g(x, y) \) with the property that \( F(x, y, g(x, y)) = 0 \).
- In general only part of a surface can be nicely parametrized. Non-uniqueness.

The definition of a surface.

- We would like a definition of a surface that as independent of representation as possible. The method of choice is: local parametrizations.
- A set of points \( S \subseteq \mathbb{R}^3 \) is a regular surface if for each \( p \in S \) there exists an open neighbourhood \( V \subseteq \mathbb{R}^3 \) containing \( p \), an open neighbourhood \( U \subseteq \mathbb{R}^2 \) and a parametrization \( \sigma : U \to V \cap S \) such that:
  1. \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \) is differentiable (i.e. each \( \sigma^i : U \to \mathbb{R} \) is a smooth function).
  2. \( \sigma \) is invertible (as a map from the parameter domain onto its image) with continuous inverse. I.e. there is a function \( \sigma^{-1} : V \cap S \to U \) such that \( \sigma \circ \sigma^{-1} = id_{V \cap S} \) and \( \sigma^{-1} \cap \sigma = id_U \); and also \( \sigma^{-1} \) is the restriction to \( V \cap S \) of a continuous function on an open neighbourhood \( W \subseteq \mathbb{R}^3 \) containing \( V \cap S \) onto \( U \).
  3. For every \( q \in U \), the differential \( D\sigma_q \) is injective.

Proof that the sphere is a regular surface by writing it as the union of six graphs over the coordinate planes. What happens at the edges of the coordinate charts?

Another example where the coordinates are differentiable at \( q \) but \( D\sigma_q \) is non-injective: the sphere in polar coordinates.

Example: graphs are regular surfaces.

Example: inverse images of a regular values are regular surfaces, again is based on the ImpFT.

- Here we have \( F(p) = 0 \) and \( DF_p \neq 0 \) meaning \( \exists i \) so that \( \frac{\partial F(p)}{\partial x^i} \neq 0 \).
- W.l.o.g. \( i = n \) so we get from the ImpFT the local solution \( x^n = g(x^1, \ldots, x^{n-1}) \) so that \( F(x^1, \ldots, x^{n-1}, g(x^1, \ldots, x^{n-1})) = 0 \).
- Now \( F^{-1}(0) \) near \( p \) projects down onto an open set \( U \) in the \((x^1, \ldots, x^{n-1})\)-plane and is equal to the graph \( \{(x^1, \ldots, x^{n-1}, g(x^1, \ldots, x^{n-1})) : (x^1, \ldots, x^{n-1}) \in U \} \). Thus it’s a surface!
Geometry versus topology.

- Explain this dichotomy.
- Euler characteristic.

The tangent space of a surface.

- Curves in a surface. The coordinate curves. Tangent vectors to a surface.
- Let $\sigma : U \subseteq \mathbb{R}^2 \rightarrow V \cap S \subseteq \mathbb{R}^3$ be a parametrization of a subset of a surface $S$ and let $p \in S$ such that $p = \sigma(u)$ for some $u \in U$. The tangent plane $T_pS$ defined as $\text{Image}(D\sigma_u) \subseteq T_{\sigma(u)}\mathbb{R}^3$.
- The previous definition depends on the parametrization $\sigma$. What if we change parametrizations? Do we get the same tangent space? Yes we do! Do change-of-parameters calculation.
- This is an example of a general principle of differential geometry: to define a geometric concept such as the tangent plane rigorously, we can use a parametrization; but then we must show independence of the particular parametrization chosen.
- Basis for the tangent space. This is NOT a geometric concept.
- Tangent space of a graph and of a level set.