1. The First Variation of Area

The mean curvature of a surface $S$ is the gradient of surface area. I.e. the surface area of $S$ decreases fastest when it is deformed by $H$ in the unit normal direction. To see this: let $\phi : U \to \mathbb{R}^3$ parametrize $S$ and let $f : U \to \mathbb{R}$ be a function. Then $\phi_{\varepsilon} := \phi + \varepsilon f \cdot N$ parametrizes a small deformation of $S$ when $\varepsilon$ is sufficiently small. Here $N$ is the unit normal vector field — note that a large class of nearby surfaces can be parametrized in this way (think about this!). Denote the deformed surface by $S_{\varepsilon}$.

We’d like to differentiate the quantity $(\text{Area})(S_{\varepsilon})$ and see what comes out. Let $g_{\varepsilon}(u) := [D\phi_{\varepsilon}]_u^T[D\phi_{\varepsilon}]_u$ and $g := g_0$. Then we can express the area of $\phi_{\varepsilon}(U)$ as

$$\text{Area}(\phi_{\varepsilon}(U)) = \int_U \sqrt{\det(g_{\varepsilon}(u))} du.$$ 

To differentiate this, we’ll need a formula for the derivative of a determinant. The formula we’ll use is a “standard” result but that you may not have seen yet. Suppose $A_{\varepsilon}$ is a differentiable family of invertible matrices. Then

$$\frac{d}{d\varepsilon} \det(A_{\varepsilon}) = \det(A_{\varepsilon}) \text{Tr}\left(A_{\varepsilon}^{-1} \frac{dA_{\varepsilon}}{d\varepsilon}\right).$$

A way to re-derive this formula in case you forget is to assume $A_{\varepsilon}$ is diagonal and apply the product rule to the determinant of $A_{\varepsilon}$, which is the product of the eigenvalues of $A_{\varepsilon}$. (The actual proof is close to this.) Now we have

$$\left.\frac{d}{d\varepsilon} \text{Area}(\phi_{\varepsilon}(U))\right|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_U \sqrt{\det(g_{\varepsilon}(u))} du\bigg|_{\varepsilon=0}

= \frac{1}{2} \int_U \text{Tr}\left(g^{-1} \frac{dg_{\varepsilon}(u)}{d\varepsilon}\right)\bigg|_{\varepsilon=0} \sqrt{\det(g(u))} du.$$

To finish, we need to differentiate $g_{\varepsilon}$. Recall that $[g_{\varepsilon}]_{ij} = \langle E_i(\varepsilon), E_j(\varepsilon) \rangle$ where

$$E_i(\varepsilon) = \frac{\partial \phi_{\varepsilon}}{\partial u^i} = E_i(0) + \varepsilon \frac{\partial f}{\partial u^i} N + \varepsilon f(u) \frac{\partial N}{\partial u^i}$$

are the coordinate vector fields. Hence

$$[g_{\varepsilon}]_{ij} = [g_0]_{ij} + \varepsilon f\left(\langle E_i(0), \frac{\partial N}{\partial u^j}\rangle + \langle \frac{\partial N}{\partial u^i}, E_j(0)\rangle\right) + O(\varepsilon^2) = [g_0]_{ij} + 2\varepsilon f A_{ij} + O(\varepsilon^2)$$

by the self-adjointness of the second fundamental form $A$ of $S$ at $u$ in the parameter domain $U$. We’ve kept track of $g_{\varepsilon}$ up to first order in $\varepsilon$ only because we’ll eventually take the derivative and then set $\varepsilon = 0$. In fact, we get

$$\left.\frac{d}{d\varepsilon} \text{Area}(\phi_{\varepsilon}(U))\right|_{\varepsilon=0} = \int_U f(u) \text{Tr}([g(u)]^{-1} A(u)) \sqrt{\det(g(u))} du = \int_U f(u) H(u) \sqrt{\det(g(u))} du.$$ 

There’s a small technicality at work here: the mean curvature is actually the sum of the eigenvalues of the shape operator $T$ which we defined indirectly by the formula $\langle T(E_i), E_j \rangle = A_{ij}$. As mentioned in a previous document, the eigenvalues of $T$ are not equal to those of $A$ unless $\{E_i\}$ is an orthonormal basis. But we can show that the eigenvalues of $T$ are equal to those of $g^{-1} A$. The matrix $g^{-1}$ corrects for the non-orthonormality of the basis.