CS 468

Differential Geometry for Computer Science

Lecture 9 — Intrinsic Geometry
Outline

From last lecture:

- The second fundamental form as extrinsic curvature.

Moving forward:

- The induced metric of a surface.
- Geodesics and length-minimizing curves.

Next time:

- The connection between the induced metric and geodesics.
Local Shape of a Surface

**Example:** Let $S$ be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Without loss of generality, we can assume $f$ vanishes to first order at $(0, 0)$.

**Then:** The second fundamental form at $(0, 0)$ is

$$A(0,0) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \Bigg|_{\text{evaluated at } (0,0)}$$

And we can characterize the origin via the eigenvalues of $A(0,0)$ as

- **Elliptic** — both $> 0$ or both $< 0$
- **Hyperbolic** — one of each sign
- **Parabolic** — one is zero,
- **Planar** — both are zero
- **Umbilic** — both are equal
Interpretation of the Mean Curvature

The mean curvature is the gradient of surface area.

- I.e. the area of the surface decreases fastest when it is deformed in the $H\mathbf{n}$ direction.

To see this:

- Let $\phi : U \rightarrow \mathbb{R}^3$ parametrize $S$ and let $f : U \rightarrow \mathbb{R}$ be a function. Then $\phi_\varepsilon := \phi + \varepsilon f \cdot N$ parametrizes a deformation of $S$.
- Finally, let $g_\varepsilon(u) := [D\phi_\varepsilon]_u^\top [D\phi_\varepsilon]_u$ and $g := g_0$. Now:

$$\frac{d}{d\varepsilon} \left. \text{Area}(\phi_\varepsilon(U)) \right|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left. \int_U \sqrt{\det(g_\varepsilon(u))} \, du \right|_{\varepsilon=0}$$

$$= \frac{1}{2} \int_U \text{Tr} \left( g^{-1} \frac{dg_\varepsilon(u)}{d\varepsilon} \bigg|_{\varepsilon=0} \right) \sqrt{\det(g(u))} \, du$$

$$= -\int_U H(u)f(u) \sqrt{\det(g(u))} \, du$$
The Induced Metric

Observation: Let $\phi : U \rightarrow \mathbb{R}^3$ parametrize a surface $S$. The object

$$g := [D\phi_u]^\top D\phi_u \quad \text{for } u \in U$$

has appeared quite often. What is the interpretation of $g$?

Definition: The object $g$ is the induced metric of $S$.

- Let $E_i := \frac{\partial \phi}{\partial u^i}$ be the tangent vectors of $S$ at $\phi(u)$.
- Then the components are $g_{ij} = E_i^\top E_j = \langle E_i, E_j \rangle$.
- Therefore the induced metric of a surface is the restriction of the Euclidean inner product to $T_{\phi(u)}S$, pulled back to $U$ via $\phi$.
- A parametrization gives you a representation of the intrinsic metric in the parameter plane as a matrix (actually a (2,0)-tensor).
Covariance

A scalar quantity defined on a surface $S$ is “geometric” if its value computed w.r.t. any parametrization is always the same.

A different property holds for vector or tensor quantities:

- The components of a “geometric” vector quantity computed w.r.t. two different parametrizations can be different.
- This is because the basis used to represent the quantity changes as well, and this must be taken into account.
- So we have transformation formulas for passing from one set of components to the other.
- This is called covariance.
Covariance of the Metric Tensor

Let $\phi: \mathcal{U} \to \mathbb{R}^3$ and $\psi: \mathcal{V} \to \mathbb{R}^3$ both parametrize $S$ with $\phi(u) = \psi(v) = p \in S$. We get:

- $e_i := [0 \ldots 1 \ldots 0]^\top$ are the standard basis vectors in $\mathcal{U}$.
- $f_i := [0 \ldots 1 \ldots 0]^\top$ are the standard basis vectors in $\mathcal{V}$.
- $E_i := \frac{\partial \phi}{\partial u^i} = D\phi_u \cdot e_i$ and $F_i := \frac{\partial \psi}{\partial v^i} = D\psi_v \cdot f_i$ are bases for $T_p S$.

Then:

$$F_i = \frac{\partial \psi}{\partial v^i} = \frac{\partial \phi \circ \phi^{-1} \circ \psi}{\partial v^i} = \sum_j \frac{\partial [\phi^{-1} \circ \psi]^j}{\partial v^i} \frac{\partial \phi}{\partial u^j} = \sum_j \frac{\partial u^j}{\partial v^i} E_j$$

And

$$\langle F_k, F_\ell \rangle = \left\langle \sum_i \frac{\partial u^i}{\partial v^k} E_i, \sum_j \frac{\partial u^j}{\partial v^\ell} E_j \right\rangle = \sum_{ij} \frac{\partial u^i}{\partial v^k} \frac{\partial u^j}{\partial v^\ell} g_{ij}$$
The Geodesic Equation

Question: What is the shortest path between \( p, q \) in a surface \( S \)?

Fact: We can find an equation satisfied by the shortest path.

- Let \( \gamma : I \rightarrow S \) be the shortest path and \( \gamma_\varepsilon \) a variation with variation vector field \( V \) that is tangent to \( S \). Then

\[
0 = \frac{d}{d\varepsilon} \text{Length}(\gamma_\varepsilon) \bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_I \|\dot{\gamma}_\varepsilon(t)\| \, dt \bigg|_{\varepsilon=0} \quad \forall \text{ variations}
\]

- From homework, we know that this implies

\[
0 = \langle \vec{k}_\gamma, V \rangle \quad \forall \text{ variations} \quad \Leftrightarrow \quad \vec{k}_\gamma \perp S
\]

Definition: Any curve satisfying this equation is called a geodesic.
The Geodesic Exponential Map

We’ll see that the geodesic equation is a second-order ODE for \( \gamma \). Thus there exists a unique local solution for every choice of

\[
p := \gamma(0) \in S \quad \text{and} \quad X := \dot{\gamma}(0) \in T_p S
\]

Definition: The assignment of \((p, X)\) to a solution at distance one is called the geodesic exponential map and is denoted

\[
\exp_p : B_\varepsilon(0) \subseteq T_p M \rightarrow M
\]

\[
\exp_p(X) := \begin{bmatrix}
\text{one unit of arc-length along the geodesic } \gamma \\
\text{with } \gamma(0) = p \text{ and } \dot{\gamma}(0) = X
\end{bmatrix}
\]

Note: The geodesic itself is given by \( \gamma(t) = \exp_p(tX) \).
Geodesics Locally Minimize Length

Two preliminary results...

Proposition: It is easy to see that $[D \exp_p]_0 = id$. Hence $\exp_p$ is a diffeomorphism near the origin in $T_p M$.

Proposition: ("Gauss lemma") Let $v, w \in T_v(T_p S)$. Then

$$\langle [D \exp_p]_v(v), [D \exp_p]_v(w) \rangle = \langle v, w \rangle$$

An important consequence...

Theorem: Geodesics locally minimize length: if $\gamma$ is a sufficiently short geodesic and $c$ is a curve with the same endpoints as $\gamma$, then

$$\text{Length}(\gamma) \leq \text{Length}(c)$$

with equality if and only if $\gamma = c$. 
Hopf-Rinow Theorem

Some facts about geodesics:

- Length-minimizing curves are geodesics.
- Short geodesics are length-minimizing.
- There exist long geodesics that are not length-minimizing.

Next: We turn $S$ into a metric space with distance function

$$d(p, q) := \inf_{\gamma \text{ from } p \text{ to } q} \text{Length}(\gamma)$$

Then $d$ is continuous and satisfies the triangle inequality.

Hopf-Rinow Theorem: If exp is globally defined then any two points $p, q$ can be connected by a geodesic with length $d(p, q)$. 