1. The Second Fundamental Form and the Shape Operator

We defined the differential of the Gauss map of a surface $S$ at $p \in S$ as the linear mapping $Dn_p : T_pS \to T_pS$. Another name for this is the shape operator (actually, $-Dn_p$ is the shape operator). Associated to the shape operator is the self-adjoint quadratic form $A_p(V,W) := -\langle Dn_p(V), W \rangle$ called the second fundamental form. A possible point of confusion from lecture today concerns the principal curvatures and directions — what matrix are they the eigenvalues and eigenvectors of?

Here is an explanation. Let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with associated quadratic form $Q(V,W) := \langle M(V), W \rangle$. Let’s assume that $M$ is symmetric and so $Q$ is self-adjoint. Define $k_{\min} := \min_{\| V \|=1} Q(V,V)$ and $k_{\max} := \max_{\| V \|=1} Q(V,V)$.

Then both $k_{\min}$ and $k_{\max}$ are eigenvalues of $M$. Let $V_{\min}$ and $V_{\max}$ be the associated eigenvectors. Then $V_{\min} \perp V_{\max}$ and can be chosen of unit length. This holds true even when $k_{\min} = k_{\max}$; now the eigenvalues are degenerate and any orthonormal vectors will do! Next, it is the case that

$$\text{Tr}(M) = k_{\min} + k_{\max} \quad \text{and} \quad \det(M) = k_{\min} \cdot k_{\max}.$$ 

To actually compute these quantities, we need to choose a basis. Note that the matrix entries of $M$ with respect to a basis $E_1, E_2$ are defined as the coefficients in the expansion $M(E_i) := \sum_j M_{ij} E_j$. Therefore the matrix entries satisfy $M_{ij} = \langle M(E_i), E_j \rangle = Q(E_i, E_j) = Q_{ij}$ if and only if the basis is orthonormal. In this case

$$k_{\min} + k_{\max} = Q_{11} + Q_{22} \quad \text{and} \quad k_{\min} \cdot k_{\max} = Q_{11}Q_{22} - Q_{12}^2.$$ 

Otherwise, let $g = \left( \begin{array}{cc} \| E_1 \|^2 & \langle E_1, E_2 \rangle \\ \langle E_1, E_2 \rangle & \| E_2 \|^2 \end{array} \right)$ and then one can show that

$$k_{\min} + k_{\max} = \sum_{ij} [g^{-1}]_{ij} Q_{ij} \quad \text{and} \quad k_{\min} \cdot k_{\max} = \frac{Q_{11}Q_{22} - Q_{12}^2}{\det(g)}.$$ 

2. Local “Shape” of a Surface

A nicer picture. The picture I drew on the board for explaining the relation between the second fundamental form $A_p$ of a surface $S$ at $p$ and the geodesic curvature of curves on $S$ passing through $p$ wasn’t very good. Here is a better picture.

I’m drawing $S$ together with a vector $X \in T_pS$ and a curve passing through $p$ in direction $X$. I’ve obtained $c$ by intersecting $S$ with the plane passing through $p$ spanned by $X$ and the normal vector $N_p$. I’ve also drawn a circle in this plane that makes second order contact with the curve $c$ at $p$. This circle has radius equal to one over the geodesic curvature $k_c(0)$; and by our formula, we also know that $k_c(0) = A_p(X,X)$. 

Classification of surface points by their curvature. From your homework assignment, we know that every surface is locally the graph of a function over its tangent plane. So without loss of generality, we can analyze the second fundamental form in the following setting. Let $S := \{(x,y, f(x,y)) : (x,y) \in \mathbb{R}^2 \}$ where $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function with $f(0,0) = 0$ and $\frac{\partial f(0,0)}{\partial x} = \frac{\partial f(0,0)}{\partial y} = 0$. You also know from your homework assignment that the tangent vectors where $E_1 = (1,0,0)^T$ and $E_2 = (0,1,0)^T$ while the second fundamental form of $S$ there is

$$[A_0]_{ij} = -\frac{\partial^2 f(0,0)}{\partial x^i \partial x^j}$$

Moreover, we know from Taylor’s theorem that

$$f(x, y) = \frac{1}{2}(x,y)D^2f(0,0)(x,y)^\top + \mathcal{O}((\|(x,y)\|)3) = -\frac{1}{2}A_0((x,y)^\top, (x,y)^\top) + \mathcal{O}((\|(x,y)\|)3).$$

Hence if $A_0$ is non-zero as a quadratic form, then $A_0$ characterizes the local shape of $S$ near the origin. That is, we can classify the origin as one of several different types:

- **The origin is an elliptic point** if either $k_{\min} > 0$ and $k_{\max} > 0$, or $k_{\min} < 0$ and $k_{\max} < 0$.
- **It is a hyperbolic point** if $k_{\min} < 0$ and $k_{\max} > 0$
- **It is a parabolic point** if one of $k_{\min} = 0$ or $k_{\max} = 0$.
- **It is a planar point** if $k_{\min} = k_{\max} = 0$.
- **It is an umbilic point** if $k_{\min} = k_{\max}$. The key feature here is that the principal directions are not uniquely defined.

We can see examples of each kind of point by choosing different functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. For instance, we can get examples of the first three kinds (and the last kind) by choosing $f(x,y) = k_{\min}x^2 + k_{\max}y^2$ which is either a paraboloid (up or down) or a hyperboloid or a degenerate quadratic form depending on the signs of the principal curvatures and whether one of them is zero or not. We get an example of the fourth kind by choosing $f(x,y) = ax + by$ — in other words $S$ is a plane.

### 3. Interpretations of the Mean and Gauss Curvatures

We’ll need this material for Wednesday’s lecture. The results will be stated here — and we’ll discuss the proof of these results briefly next Monday.

**Mean curvature as first variation of area.** Let $S$ be an orientable surface and consider a deformation of $S$ constructed in the following way. Choose a function $f : S \rightarrow \mathbb{R}$ and a small number $\varepsilon > 0$ and displace each $p \in S$ by an amount $\varepsilon f(p)$ in the normal direction at $p$. In other words $p_{\text{displaced}} := p + \varepsilon f(p)N_p$. The new surface is $S_\varepsilon := \{p_{\text{displaced}} : p \in S\}$.

Now as $S$ deforms into $S_\varepsilon$, its surface area changes. We will see that

$$\left.\frac{d}{d\varepsilon}\text{Area}(S_\varepsilon)\right|_{\varepsilon = 0} = -\int_S f(p)H(p)d\text{Area}(p).$$

In other words, the first order change in the area is given by integration against the mean curvature. This also means that if $f(p) = H(p)$ then the surface area decreases the fastest. In other words, we can interpret the mean curvature as the gradient of the surface area functional.

**Gauss curvature in terms of the Gauss map.** This time we keep the surface $S$ fixed and consider small balls about a point $p \in S$. where $K(p) \neq 0$. Let $\varepsilon > 0$ be such that $K$ does not change sign in $B_\varepsilon(p)$. Then if $n$ denotes the Gauss map, we will show that

$$K(p) = \lim_{\varepsilon \rightarrow 0} \frac{\text{Area}(n(B_\varepsilon(p)))}{\text{Area}(B_\varepsilon(p))}$$