The unit normal vector of a surface.

- The normal vector of a surface. Is this geometric?
- The normal line is geometric but the normal direction may not be. Non-orientable surfaces.
- Normal vector of a parametrized surface, graph and level set.

Surface Area.

- Setting up the Riemann sum that yields the surface area of a surface.
- Area of infinitesimal coordinate rectangle and the Riemannian area form.
- Independence of parametrization of the area integral.

The Gauss map.

- Let $S$ be an orientable surface with unit normal vector field $n_p$ at each $p \in S$. The Gauss map of $S$ is the mapping $N: S \to \mathbb{S}^2$ given by $N(p) := n_p$. Here we view the unit normal vector at a given $p \in S$ as a vector in $\mathbb{R}^3$ of length one and thus a point in $\mathbb{S}^2$.
- The Gauss map of a differentiable surface is itself differentiable. Thus we can study its differential $DN_p: T_pS \to T_{n_p}\mathbb{S}^2$.
  - We can define the differential rigorously as follows. Let $V_p$ be a tangent vector to $S$ at $p$ generated by a curve $c: (-\varepsilon, \varepsilon) \to S$. In other words, $c(0) = p$ and $\frac{dc}{dt} \bigg|_{t=0} = V_p$. Then $DN_p(V_p) := \frac{d}{dt}N_p(c(t)) \bigg|_{t=0}$. This is well-defined because we can show that the choice of curve doesn’t matter.
  - Note that because $N(p) \in \mathbb{S}^2$ for each $p$ it really is the case that $DN_p(V)$ is tangent to $\mathbb{S}^2$ for any vector $V \in T_pS$.
  - Prove this by differentiating $\|N(c(t))\|^2 = 1$ where $c: [-1, 1] \to S$ is a curve in $S$.
- Some examples. Gauss map of parametrized surface, level set and graph.

Definition of the second fundamental form.

- Since $T_pS$ and $T_{n(p)}\mathbb{S}^2$ are parallel planes (they’re both perpendicular to $n_p$), we can consider the differential of the Gauss map as a map $DN_p: T_pS \to T_pS$.
- Proposition: viewed in this way, $DN_p$ is self-adjoint with respect to the Euclidean metric of $\mathbb{R}^3$ restricted to $T_pS$.
- Definition: the second fundamental form at $p \in S$ is the bilinear form $A_p: T_pS \times T_pS \to \mathbb{R}$ defined by $A_p(V, W) := -\langle DN_p(V), W \rangle$ for any $V, W \in T_pS$.
- $A_p(V, W)$ measure the projection onto $W$ of the rate of change of $N_p$ in the $V$-direction.
- Example calculations.
The second fundamental form as extrinsic curvature.

- Let \( c : [-1, 1] \to S \) be a curve in \( S \) with \( c(0) = p \). Then the geodesic curvature vector of \( c \) at zero is related to the second fundamental form at \( p \) as follows: \( \langle \vec{k}_c(0), n_p \rangle = A_p(\dot{c}(0), \dot{c}(0)) \). Note this is independent of \( \ddot{c} \) or \( c(t), \dot{c}(t) \) for \( t \neq 0 \).

- Let \( V \) vary over all unit vectors in \( T_pS \). Then \( A_p(V, V) \) takes on a minimum value \( k_{\text{min}} \) and a maximum value \( k_{\text{max}} \). These are the principal curvatures of \( S \) at \( p \) and are eigenvalues of \( A_p \). The corresponding eigenvectors \( V_{\text{min}} \) and \( V_{\text{max}} \) are the principal directions of \( A_p \). Note that \( V_{\text{min}} \) and \( V_{\text{max}} \) are orthogonal.

- Mean curvature and Gauss curvature.

Local “shape” of a surface.

- Definitions of elliptic, hyperbolic, parabolic, planar or umbilic points.

- Examples of each type.

- Local characterization of the surface \( S \) at \( p \) depending its type. Proof based on Taylor series expansion in the “right” coordinate system: a neighbourhood of \( p \) is the graph of a function over \( T_pS \).

Interpretation of the Gauss curvature in terms of the Gauss map.

- Lemma: \( K(p) > 0 \) iff Gauss map locally preserves orientation; \( K(p) < 0 \) iff Gauss map locally reverses orientation.

- Proposition: Let \( p \in S \) be such that \( K(p) \neq 0 \) and let \( \varepsilon > 0 \) be such that \( K \) does not change sign in \( B_\varepsilon(p) \). Then if \( N \) denotes the Gauss map, we have

\[
K(p) = \lim_{\varepsilon \to 0} \frac{\text{Area}(N(B_\varepsilon(p)))}{\text{Area}(B_\varepsilon(p))}
\]

Interpretation of the mean curvature as first variation of area.

- Another interpretation for the second fundamental form — at least of its trace, the mean curvature.

- The calculation.