Level sets.

- Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a function and let $c \in \mathbb{R}$ be a number. The level set of $F$ with value $c$ is the set of points

$$F^{-1}(c) := \{ p \in \mathbb{R}^3 : F(p) = c \}$$

- So to find a level set, you must solve the equation $F(p) = c$ for $p = (x, y, z)$.
- Note: if there are no solutions, then $F^{-1}(c) = \emptyset$ (the empty set).

Level sets as surfaces.

- The big question: what is the geometric nature of a level set?
- Our intuition says the a level set is a surface because a level set consists of the solution of “one scalar equation in three unknowns.”
- The reasoning is: by solving the equations you should be able to express one of the unknowns as a function of the other two. In other words, you can write $z = g(x, y)$ for some function $g$, and $F(x, y, g(x, y)) = c$. Now the solution set looks like

$$\{(x, y, g(x, y)) : x, y^2 \in U \subseteq \mathbb{R}^2 \}$$

In other words, the solution set is a graph, which is a surface as we saw in class.

Exceptions.

- There are exceptions to the nice intuitive picture described above.
- For example, consider the function $F(x, y, z) := x^2 + y^2 + z^2$. The level set of $c > 0$ is a sphere of radius $\sqrt{c}$ — which is a surface. The level set of $c < 0$ is the empty set. The level set of $c = 0$ consists of the point $(0, 0, 0)$ only. In other words, $F^{-1}(0) = \{(0, 0, 0)\}$. This is not a surface.
- There are other examples where $F^{-1}(c)$ is not a surface. For instance, the level set of zero of the function $F(x, y, z) = x^2 + y^2$ is the $z$-axis, which is a line and not a surface. (Other level sets with $c > 0$ are cylinders and with $c < 0$ are the empty set.)
- Even weirder things can happen. For instance, the level set of zero of the function $F(x, y, z) := xy$ is the union of the $(y, z)$-plane and the $(x, z)$-plane which is not a surface in the neighbourhood of the $z$-axis. (Draw this object!)
- Much, much weirder things can happen.

Regular values.

- We would like to characterize when a level set is a surface. We will need the concept of a regular value.
- Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. A number $c \in \mathbb{R}$ is a regular value for $F$ if the derivative matrix of $F$ (which is a $1 \times 3$ matrix in this case) does not vanish anywhere on the level set $F^{-1}(c)$.
- I.e. $c$ is a regular value for $F$ if $DF_p = (\frac{\partial F(p)}{\partial x}, \frac{\partial F(p)}{\partial y}, \frac{\partial F(p)}{\partial z}) \neq (0, 0, 0)$ for all $p \in F^{-1}(0)$. 
The inverse image of a regular value is a surface.

- Suppose \( c \) is a regular value for \( F \) and let \( p \in F^{-1}(c) \).
- Without loss of generality, we can assume that \( \frac{\partial F(p)}{\partial z} \neq 0 \).
- We now invoke the Implicit Function Theorem.

- Write \( F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R} \). Now the matrix \( D_2 F_p \) appearing in this theorem is simply the number \( \frac{\partial F(p)}{\partial z} \). So the invertibility of \( D_2 F_p \) is equivalent to \( \frac{\partial F(p)}{\partial z} \neq 0 \).
- The Implicit Function Theorem now gives us a local solution \( z = g(x, y) \) where \( g : U \to \mathbb{R} \) is a smooth function defined in a neighbourhood of \( p \).
- Now \( F^{-1}(0) \) near \( p \) can be parametrized with the help of \( g \). That is, we can write \( F^{-1}(0) \) near \( p \) as \( \{ (x, y, g(x, y)) : (x, y) \in U \} \).
- In other words, \( F^{-1}(0) \) near \( p \) is the graph of \( g \). This is a regular surface!

A nice formula.

- We can relate the derivatives of \( g \) to the derivatives of \( F \) using the chain rule.
- We have \( F(x, y, g(x, y)) = c \) so for instance
  
  \[
  0 = \frac{\partial F(x, y, g(x, y))}{\partial x} = \frac{\partial F}{\partial x}(x, y, g(x, y)) + \frac{\partial F}{\partial z}(x, y, g(x, y)) \cdot \frac{\partial g}{\partial x}
  \]

- By isolating \( \frac{\partial g}{\partial x} \) we obtain the formula
  
  \[
  \frac{\partial g}{\partial x} = -\frac{\frac{\partial F}{\partial x}(x, y, g(x, y))}{\frac{\partial F}{\partial z}(x, y, g(x, y))}
  \]

  which is a sensible mathematical expression so long as \( \frac{\partial F}{\partial z} \neq 0 \) which is certainly true sufficiently close to \( p \).
- A similar formula holds for \( \frac{\partial g}{\partial y} \).