<review>
**Review**

Function of time:

\[ s(t) = \int_{0}^{t} \| \gamma' \| \, dt \]

Function of distance:

\[ \gamma(s) \]
The Geometric Object

Trace of the curve

\{ \gamma(t) : t \in \mathbb{R} \}
Unit Tangent

Intuition: Why unit length?
Take $v(t) : \mathbb{R} \to \mathbb{R}^n$ such that
\[ \|v(t)\| = 1 \quad \forall t. \]
Show $\langle v(t), v'(t) \rangle = 0 \quad \forall t.$
Frenet Frame

\[ \frac{dT}{ds} = \kappa N \]
\[ \frac{dN}{ds} = -\kappa T + \tau B \]
\[ \frac{dB}{ds} = -\tau N \]
Old-School Approach

$F(0, 0, 1) \quad F(0, 1, 1) \quad F(t, t, t) = f(t) \quad F(1, 1, 1) = f(1)$

$F(0, 0, 0) = f(0) \quad F(0, 1, 1) = f(1)$

Piecewise smooth approximations
Old-School Approach

Piecewise smooth approximations

\[ F(0, 0, 0) = f(0) \]
\[ F(0, 1, 1) \]
\[ F(t, t, t) = f(t) \]
\[ F(1, 1, 1) = f(1) \]

Calculus applies (except at joints!)

Piecwise smooth approximations
What is the arc length of a cubic Bézier curve?

\[ s = \int_{t_0}^{t_1} \sqrt{\gamma'_x^2 + \gamma'_y^2 + \gamma'_z^2} \, dt \]
Question

What is the arc length of a cubic Bézier curve?

\[ s = \int_{t_0}^{t_1} \sqrt{\gamma_x'^2 + \gamma_y'^2 + \gamma_z'^2} \, dt \]

Hint: It’s usually impossible.
Sad fact:
Closed-form expressions rarely exist. When they do exist, they usually are messy.
\{\text{Bézier curves}\} \subsetneq \{\gamma : \mathbb{R} \rightarrow \mathbb{R}^3\}
Only Approximations Anyway

\{ \text{Bézier curves} \} \subsetneq \{ \gamma : \mathbb{R} \to \mathbb{R}^3 \}
Equally Good Approximation

Piecewise linear
Big Problem

Boring differential structure

$\kappa = 0$

$\kappa = \infty$
Finite Difference Approach

\[ f'(x) \approx \frac{1}{h}(f(x + h) - f(x)) \]

THEOREM: As \( \Delta h \to 0 \), [insert statement].
Theorem: As $\Delta h \to 0$, 

$\frac{f(x + h) - f(x)}{h} \neq 0$
Two Key Considerations

- Convergence to continuous theory
- Discrete behavior
Today’s Goal

Examine discrete theories of differentiable curves.
Examine discrete theories of differentiable curves.
Gauss Map

Normal map from curve to $S^1$
Signed Curvature on Plane Curves

\[ T(s) = (\cos \theta(s), \sin \theta(s)) \]

\[ T'(s) = \theta'(s)(-\sin \theta(s), \cos \theta(s)) \equiv \kappa(s)N(s) \]
Turning Numbers

+1  -1  +2  0
\[ \theta'(s) \equiv \kappa(s) \]
\[ \Downarrow \]
\[ \Delta \theta = \int_{s_0}^{s_1} \kappa(s) \, ds \]
Turning Number Theorem

\[ \int \kappa(s) \, ds = 2\pi k \]

A “global” theorem!
Discrete Gauss Map
Discrete Gauss Map

Edges become points
Discrete Gauss Map

Vertices become arcs
Key Observation

\[ \sum_{i} \theta_i = 2\pi k \]
What’s Going On?

Total change in curvature
What’s Going On?

\[ \theta = \int_{\Gamma} \kappa \, ds \]

Total change in curvature
What’s Going On?

\[ \theta = \int_{\Gamma} \kappa \, ds \]

\[ \kappa \approx \frac{\theta}{\ell_1 + \ell_2} \]

Total change in curvature
Interesting Distinction

\[ \kappa_1 \ll \kappa_2 \]

Same integrated curvature
Interesting Distinction

\[ \kappa_1 \ll \kappa_2 \]

Same integrated curvature
\[ \theta = \int_{\Gamma} \kappa \, ds \]

Integrated quantity

Total change in curvature

Dual cell
Discrete Turning Angle Theorem

\[ \int_{\Gamma} \kappa \, ds = \sum_i \int_{\Gamma_i} \kappa \, ds = \sum_i \theta_i = 2\pi k \]
Alternative Definition

$-\kappa N$ decreases length the fastest.

Homework
\[ \nabla L = 2N \sin \frac{\theta}{2} \]
For Small $\theta$

\[
2 \sin \frac{\theta}{2} \approx 2 \cdot \frac{\theta}{2} = \theta
\]

Same behavior in the limit

http://en.wikipedia.org/wiki/Taylor_series
Does discrete curvature converge in limit?

Yes!
Remaining Question

Does discrete curvature converge in limit?

Questions:
- Type of convergence?
- Sampling?
- Class of curves?

Yes!
Different discrete behavior

Same convergence
Curves in 3D
Frenet Frame

\[
\begin{align*}
\frac{dT}{ds} &= \kappa N \\
\frac{dN}{ds} &= -\kappa T + \tau B \\
\frac{dB}{ds} &= -\tau N
\end{align*}
\]

http://upload.wikimedia.org/wikipedia/commons/6/6f/Frenet.png
Potential Discretization

\[t_j = \frac{p_{j+1} - p_j}{\|p_{j+1} - p_j\|}\]

\[b_j = t_{j-1} \times t_j\]

\[n_j = b_j \times t_j\]

Discrete Frenet frame

\[t_k = R(b_k, \theta_k)t_{k-1}\]

\[b_{k+1} = R(t_k, \phi_k)b_k\]

“Bond and torsion angles”
(derivatives converge to \(\kappa\) and \(\tau\), resp.)

Discrete frame introduced in:
The resultant electric moment of complex molecules

Structure Determination of Membrane Proteins Using Discrete Frenet Frames
and Solid State NMR Restraints
Achuthan and Quine
Structure Determination of Membrane Proteins Using Discrete Frenet Frames and Solid State NMR Restraints

Achuthan and Quine

Transfer Matrix

\[
\begin{pmatrix}
t_{i+1} \\
n_{i+1} \\
b_{i+1}
\end{pmatrix} = \mathcal{R}_{i+1,i}
\begin{pmatrix}
t_i \\
n_i \\
b_i
\end{pmatrix}
\]

*Discrete construction that works for fractal curves and converges in continuum limit.*

Discrete Frenet Frame, Inflection Point Solitons, and Curve Visualization with Applications to Folded Proteins
Hu, Lundgren, and Niemi
*Physical Review E 83* (2011)
Frenet Frame: Issue

\[ \frac{dT}{ds} = \kappa N \]
\[ \frac{dN}{ds} = -\kappa T + \tau B \]
\[ \frac{dB}{ds} = -\tau N \]

\( \kappa = 0? \)
Segments Not Always Enough

Discrete Elastic Rods
Bergou, Wardetzky, Robinson, Audoly, and Grinspun
SIGGRAPH 2008

http://www.cs.columbia.edu/cg/rods/
Adapted Framed Curve

\{ t = \gamma', m_1, m_2 \} 

Material frame

Normal part encodes twist

http://www.cs.columbia.edu/cg/rods/
Bending Energy

\[ E_{\text{bend}}(\Gamma) = \frac{1}{2} \int_{\Gamma} \alpha \kappa^2 \, ds \]

Punish turning the steering wheel

\[ \kappa n = t' \]

\[ = (t' \cdot t)t + (t' \cdot m_1)m_1 + (t' \cdot m_2)m_2 \]

\[ = (t' \cdot m_1)m_1 + (t' \cdot m_2)m_2 \]

\[ \equiv \omega_1 m_1 + \omega_2 m_2 \]
Bending Energy

\[
E_{bend}(\Gamma) = \frac{1}{2} \int_{\Gamma} \alpha(\omega_1^2 + \omega_2^2) \, ds
\]

Punish turning the steering wheel

\[
\kappa n = t' = (t' \cdot t)t + (t' \cdot m_1)m_1 + (t' \cdot m_2)m_2 = (t' \cdot m_1)m_1 + (t' \cdot m_2)m_2 = \omega_1 m_1 + \omega_2 m_2
\]
Twisting Energy

\[ E_{\text{twist}}(\Gamma) = \frac{1}{2} \int_{\Gamma} \beta m^2 \, ds \]

Punish non-tangent change in material frame

\[ m \equiv m'_1 \cdot m_2 \]

\[ = \frac{d}{dt} (m_1 \cdot m_2) - m_1 \cdot m'_2 \]

\[ = -m_1 \cdot m'_2 \]
Twisting Energy

\[ E_{\text{twist}}(\Gamma) = \frac{1}{2} \int_{\Gamma} \beta m^2 \, ds \]

Punish non-tangent change in material frame

\[ m \equiv m'_1 \cdot m_2 \]

\[ = \frac{d}{dt}(m_1 \cdot m_2) - m_1 \cdot m'_2 \]

\[ = -m_1 \cdot m'_2 \]

Swapping \( m_1 \) and \( m_2 \) does not affect \( E_{\text{twist}} \)!
THERE IS MORE THAN ONE WAY TO FRAME A CURVE

RICHARD L. BISHOP

The Frenet frame of a 3-times continuously differentiable (that is, $C^3$) non-degenerate curve in euclidean space has long been the standard vehicle for analysing properties of the curve invariant under euclidean motions. For arbitrary moving frames, that is, orthonormal basis fields, we can express the derivatives of the frame with respect to the curve parameter in terms of the frame itself, and due to orthormal-normality the coefficient matrix is always skew-symmetric. Thus it generally has three nonzero entries. The Frenet frame gains part of its special significance from the fact that one of the three derivatives is always zero. Another feature of the Frenet frame is that it is adapted to the curve: the members are either tangent to or perpendicular to the curve. It is the purpose of this paper to show that there are other frames which have these same advantages and to compare them with the Frenet frame.

1. Relatively parallel fields. We say that a normal vector field $M$ along a curve is relatively parallel if its derivative is tangential. Such a field turns only whatever amount is necessary for it to remain normal, so it is as close to being parallel as possible without losing normality. Since its derivative is perpendicular to it, a relatively parallel normal field has constant length. Such fields occur classically in
Bishop Frame

\[ t' = \times t \]
\[ u' = \times u \]
\[ v' = \times v \]
\[ \equiv \kappa b \]

Most relaxed frame

Darboux vector

http://www.cs.columbia.edu/cg/rods/
\[ m_1 = u \cos \theta + v \sin \theta \]
\[ m_2 = -u \sin \theta + v \cos \theta \]

\[ E_{\text{twist}}(\Gamma) = \frac{1}{2} \int_{\Gamma} \beta(\theta')^2 \, ds \]

Degrees of freedom for elastic energy:
- Shape of curve
- Twist angle \( \theta \)
Discrete Kirchoff Rods

- Lower index: primal
- Upper index: dual
Discrete Kirchoff Rods

\[ t^i = \frac{e^i}{\|e^i\|} \]

Tangent unambiguous on edge
Discrete Kirchoff Rods

$x_0 e^0 x_1 e^1 e^2 e^3 x_2 e^2 x_3 e^3 x_4$

$\kappa_i = 2 \tan \frac{\phi_i}{2}$

Yet another curvature!

Integrated curvature
Discrete Kirchoff Rods

Yet another curvature!

\[ \kappa_i = 2 \tan \frac{\phi_i}{2} \]

\[ (\kappa b)_i = \frac{2e^{i-1} \times e^i}{\|e^{i-1}\| \|e^i\| + e^{i-1} \cdot e^i} \]

Orthogonal to osculating plane, norm \( \kappa_i \)

Darboux vector
Bending Energy

\[ E_{\text{bend}}(\Gamma) = \frac{\alpha}{2} \sum_i \left( \frac{(\kappa b)_i}{\ell_i/2} \right)^2 \ell_i \]

\[ = \alpha \sum_i \frac{\| (\kappa b)_i \|^2}{\ell_i} \]

Can extend for natural bend

Convert to pointwise and integrate
Discrete Parallel Transport

\[ P_i(t_{i-1}) = t^i \]

\[ P_i(t_{i-1} \times t^i) = t_{i-1} \times t^i \]

- Map tangent to tangent
- Preserve binormal
- Orthogonal

\[ u^i = P_i(u_{i-1}) \]

\[ v^i = t^i \times u^i \]

http://www.cs.columbia.edu/cgi/rods/
\( m_1^i = u^i \cos \theta^i + v^i \sin \theta^i \)
\( m_2^i = -u^i \sin \theta^i + v^i \cos \theta^i \)
Discrete Twisting Energy

\[ E_{\text{twist}}(\Gamma) = \beta \sum_i \frac{(\theta^i - \theta^{i-1})^2}{\ell_i} \]
Discrete Twisting Energy

\[ E_{\text{twist}}(\Gamma) = \beta \sum_{i} \frac{(\theta^i - \theta^{i-1})^2}{l_i} \]

Note \( \theta_0 \) can be arbitrary
\omit{physics}
\omit{physics}

Worth reading!
# Extension and Speedup

## Discrete Viscous Threads

<table>
<thead>
<tr>
<th>Name</th>
<th>Institution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miklós Bergou</td>
<td>Columbia University</td>
</tr>
<tr>
<td>Basile Audoly</td>
<td>UPMC Univ. Paris 06 &amp; CNRS</td>
</tr>
<tr>
<td>Etienne Vouga</td>
<td>Columbia University</td>
</tr>
<tr>
<td>Max Wardetzky</td>
<td>Universität Göttingen</td>
</tr>
<tr>
<td>Eitan Grinspun</td>
<td>Columbia University</td>
</tr>
</tbody>
</table>

[http://www.cs.columbia.edu/cg/threads/]
Extension and Speedup

Discrete Viscous Threads

Miklós Bergou
Columbia University

Basile Audoly
UPMC Univ. Paris 06 & CNRS

Etienne Vouga
Columbia University

Max Wardetzky
Universität Göttingen

Eitan Grinspun
Columbia University

“...the first numerical fluid-mechanical sewing machine.”
One curve, three curvatures.

\[ \theta \quad 2 \sin \frac{\theta}{2} \quad 2 \tan \frac{\theta}{2} \]
Easy theoretical object, hard to use.

\[
\begin{align*}
\frac{dT}{ds} &= \kappa N \\
\frac{dN}{ds} &= -\kappa T + \tau B \\
\frac{dB}{ds} &= -\tau N
\end{align*}
\]
Proper coordinates and DOFs go a long way.

\[ m_1^i = u^i \cos \theta^i + v^i \sin \theta^i \]
\[ m_2^i = -u^i \sin \theta^i + v^i \cos \theta^i \]
Next

Surfaces

http://graphics.stanford.edu/data/3Dscanrep/stanford-bunny-cebal-ssh.jpg
http://www.stat.washington.edu/wxs/images/BUNMID.gif
Discrete Curves