CS 468

Differential Geometry for Computer Science

Lecture 2 — Curves
Definition of a curve

- A parametrized differentiable curve is a differentiable map \( \gamma : I \rightarrow \mathbb{R}^n \) where \( I = (a, b) \) is an interval in \( \mathbb{R} \).
- The parameter domain \( I \).
- The image or trace of \( \gamma \).
- The component functions of \( \gamma \).
Velocity and Acceleration

- Instantaneous velocity.
- Instantaneous acceleration.
- Constant speed curves and constant velocity curves.
- Singular points.
Examples

- Lines in space.
- Circle in $\mathbb{R}^2$.
- Helix in $\mathbb{R}^3$.
- Self-intersection — embedded vs. immersed curves.
- Curve with a kink, curve with a cusp — smooth but singular, and non-smooth parametrizations thereof.
Change of parameter

- Definition of reparametrization.
- The trace remains unchanged.
- Effect on velocity and acceleration.
Arc-length

- Arc-length is the limit of a sequence of discrete approximations.
- Derivation: let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth curve and partition $I = [t_0, t_1] \cup \cdots \cup [t_{n-1}, t_n]$ with $t_0 = a$ and $t_n = b$. Now

$$
\text{length}(\gamma([a, b])) \approx \sum_{i=1}^{n} \| \gamma(t_i) - \gamma(t_{i-1}) \|
$$

$$
= \sum_{i=1}^{n} \| \gamma(t_i^*) \Delta t_i \|
$$

$$
= \sum_{i=1}^{n} \| \dot{\gamma}(t_i^*) \| \Delta t_i
$$

$$
\xrightarrow{n \to \infty} \int_{a}^{b} \| \dot{\gamma}(t) \| dt
$$
Parameter independence of arc-length

Let $\phi : [a, b] \to [a, b]$ be a diffeomorphism with $\phi(a) = a$ and $\phi(b) = b$. Let $\tilde{\gamma}(s) := \gamma(\phi(s))$. Then

$$\text{length}(\tilde{\gamma}([a, b])) = \int_{a}^{b} \left\| \frac{d\gamma \circ \phi(s)}{ds} \right\| ds$$

$$= \int_{a}^{b} |\phi'(s)| \left\| \frac{d\gamma}{dt} \circ \phi(s) \right\| ds$$

$$= \int_{a}^{b} |\phi' \circ \phi^{-1}(t)| \left\| \frac{d\gamma(t)}{dt} \right\| \left| \frac{dt}{\phi' \circ \phi^{-1}(t)} \right|$$

$$= \int_{a}^{b} \left\| \frac{d\gamma(t)}{dt} \right\| dt$$

$$= \text{length}(\gamma([a, b]))$$
Example calculations

- Mostly no closed form for arc-lengths.
- First example: logarithmic spiral $\gamma(t) = (e^t \cos(t), e^t \sin(t))$.
- Second example: $\gamma(t)$ such that $\|\dot{\gamma}\| = \text{const}$.
- Parametrization by arc-length.
Arc-length re-parametrization

• We can re-parametrize any curve so that it is parametrized by arc-length. (Useful theoretically but hard to put into practice.)

• Let $\gamma : I \rightarrow \mathbb{R}$ be a smooth curve and define the function

$$\ell : I \rightarrow [0, \text{length}(\gamma(I))]$$

$$\ell(t) := \int_{0}^{t} \|\dot{\gamma}(x)\| \, dx$$

• Invertibility of $\ell$ when $\gamma$ is non-singular.

• Define a new parameter $s$ that satisfies $s = \ell(t)$. Define the re-parametrized version of $\gamma$, namely $\tilde{\gamma}(s) = \gamma(\ell^{-1}(s))$.

• This re-parametrized version is parametrized by arc-length.

• Example: the logarithmic spiral.
Curvature

• Definition of the geodesic curvature vector in an arbitrary parametrization — the normal component of the acceleration vector, normalized by the squared length of the tangent vector.

\[ \vec{k}_c := \frac{\vec{\ddot{c}}}{\|\dot{c}\|^2} \]

• Definition of the geodesic curvature \( k_c := \|\vec{k}_c\| \).

• In the arc-length parametrization we have \( \vec{k}_c = [\ddot{c}]^\perp \).

• Examples.
The Frenet frame

- Let $\gamma : \rightarrow \mathbb{R}^3$ be a curve, w.l.o.g parametrized by arc-length.
- We will find a choice of “moving axes best adapted to the geometry of $\gamma$.
- Let $T(s) := \dot{\gamma}(s)$.
- A point of non-zero curvature allows us to define a distinguished normal vector $N(s) := \frac{\dot{T}(s)}{\|\dot{T}(s)\|}$.
- The osculating plane at $\gamma(s)$ is spanned by $T(s), N(s)$.
- The binormal vector is $B(s) := T(s) \times N(s)$.
- The Frenet frame for $\gamma$ is $\{T(s), N(s), B(s)\}$ and is defined at each point $\gamma(s)$ where $k_\gamma(s) \neq 0$. 
The Frenet formulas

- The Frenet formulas explain the variation in the Frenet frame along $\gamma$.

\[
\dot{T}(s) = k_\gamma(s)N(s)
\]

\[
\dot{N}(s) = \langle \dot{N}(s), T(s) \rangle T(s) + \langle \dot{N}(s), N(s) \rangle N(s) + \langle \dot{N}(s), B(s) \rangle B(s)
= -k_\gamma(s)T(s) + \langle \dot{N}(s), B(s) \rangle B(s)
= -k_\gamma(s)T(s) - \tau_\gamma(s)B(s)
\]

\[
\dot{B}(s) = \langle \dot{B}(s), T(s) \rangle T(s) + \langle \dot{B}(s), N(s) \rangle N(s) + \langle \dot{B}(s), B(s) \rangle B(s)
= -\langle B(s), \dot{T}(s) \rangle T(s) + \langle \dot{B}(s), N(s) \rangle N(s)
= -k_\gamma(s)\langle B(s), N(s) \rangle T(s) - \langle N(s), \dot{B}(s) \rangle B(s)
= \tau_\gamma(s)B(s)
\]

- Here we have introduced the torsion $\tau_\gamma(s) := -\langle \dot{N}(s), B(s) \rangle$. 
A local theorem

- Locally, $k$ and $\dot{k}$ determine the amount of turning in the osculating plane.
- And $\tau$ and $k$ determine the amount of lifting out of the osculating plane into its normal direction.
A global theorem

- The Fundamental Theorem of Curves.

Suppose we are given differentiable functions $k : I \to \mathbb{R}$ with $k > 0$, and $\tau : \to \mathbb{R}$.

Then there exists a regular curve $\gamma : I \to \mathbb{R}^3$ such that $s$ is the arc-length, $k(s)$ is the geodesic curvature, and $\tau(s)$ is the torsion.

Any other curve satisfying the same conditions differs from $\gamma$ by a rigid motion.