Definition of a curve.

- Definition. A parametrized differentiable curve in $\mathbb{R}^n$ is a differentiable map $\gamma : I \to \mathbb{R}^n$ where $I = (a, b)$ is an open interval in $\mathbb{R}$. Note: $I$ can be a closed interval — now we have a curve with boundary points.
- Notation. Such a map has component functions $\gamma(t) := (\gamma_1(t), \ldots, \gamma_n(t))$. Each $\gamma_i : I \to \mathbb{R}$ is a differentiable function.
- The domain $I$ is the space where the parameter $t$ lives.
- The image of $\gamma$ is the set of points $\{ \gamma(t) : t \in I \} \subseteq \mathbb{R}^n$. It is a geometric thing called the trace of the curve. We interpret $\gamma(t)$ as the location of a particle in space at the instant of time $t$; and we interpret the trace of the curve as the path traced out by the particle as $t$ varies in $I$.
- Distinction between this kind of curve and a 1-D manifold.

Velocity and Acceleration.

- Instantaneous velocity of the particle at time $t$ is $\dot{\gamma}(t) = (\dot{\gamma}_1(t), \ldots, \dot{\gamma}_n(t))$.
- Instantaneous acceleration of the particle at time $t$ is $\ddot{\gamma}(t) = (\ddot{\gamma}_1(t), \ldots, \ddot{\gamma}_n(t))$.
- Constant speed curves; acceleration is normal to the velocity. Constant velocity curves are straight lines.
- Singular points where $\dot{\gamma} = 0$. The parametrized map can still be differentiable but the trace may not be smooth. For example: 

$$
\gamma(t) := \begin{cases} 
(e^{-1/t^2}, 0) & t > 0 \\
0 & t = 0 \\
(0, e^{-1/t^2}) & t < 0
\end{cases}
$$

Examples.

- Lines in space: $\gamma(t) = x_0 + tv$ is the line passing through $x_0$ in direction $v$.
- Circle in $\mathbb{R}^2$, helix in $\mathbb{R}^3$.
- Curve in which the trace intersects itself
- Curve with a kink, curve with a cusp — smooth (with singular point) and non-smooth parametrizations thereof (e.g. $\gamma(t) = (t^3, t^2)$ or $\tilde{\gamma}(t) = (t, t^{2/3})$).
- An exotic example. E.g. Cycloid — the motion of a point on the rim of a wheel of radius $R$ as the wheel rolls without slipping along the $x$-axis. (This is derived as follows. Let $\theta$ be the angle through which the wheel has rolled. Then the distance the point of contact with the ground has moved is equal to $R\theta$. Hence the position of the centre of the wheel has moved to $(R\theta, R)$. And the point on the edge of the wheel, originally touching the ground at $\theta = 0$ has rotated through a clockwise angle of $\theta$ measured relative to the centre of the wheel. In other words, this point is located at 

$$
\gamma(\theta) := (R\theta, R) + (R \cos(-\pi/2 - \theta), R \sin(-\pi/2 - \theta)) = (R\theta, R) - (R \sin(\theta), R \cos(\theta)) .
$$
Change of parameter.

- Definition of reparametrization: a bijective map \( \phi : J \rightarrow I \) gives you a new curve \( \tilde{\gamma} : J \rightarrow \mathbb{R}^n \) defined by \( \tilde{\gamma}(s) = \gamma(\phi(s)) \). The formula \( t = \phi(s) \) is a change of parameter.

- Note that a smooth mapping \( \phi \) between intervals is a bijection if and only if \( \phi' \) never vanishes.

- The trace remains unchanged.

- Effect on velocity and acceleration:

\[
\frac{d^2 \tilde{\gamma}(s)}{ds^2} = \frac{d^2 \gamma(\phi(s))}{ds^2} = \frac{d^2 \gamma}{dt^2} \circ \phi(s) \frac{d\phi(s)}{ds}
\]

Note length changes

\[
\frac{d^2 \gamma}{dt^2} \circ \phi(s) \left( \frac{d\phi(s)}{ds} \right)^2 + \frac{d\gamma}{dt} \circ \phi(s) \frac{d^2 \phi(s)}{ds^2}
\]

Arc length.

- Discrete approximation of the length of a differentiable curve by means of line segments; limit as segment length \( \to 0 \) yields the arc length integral.

- Derivation: let \( \gamma : [a, b] \rightarrow \mathbb{R}^3 \) be a smooth curve and partition \( I = [t_0, t_1] \cup \cdots \cup [t_{n-1}, t_n] \) with \( t_0 = a \) and \( t_n = b \). Suppose \( \gamma(t) = (x(t), y(t), z(t)) \). Now compute

\[
\text{length}(\gamma([a, b])) \approx \sum_{i=1}^{n} \| \gamma(t_i) - \gamma(t_{i-1}) \|
\]

\[
= \sum_{i=1}^{n} \left( (x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 + (z(t_i) - z(t_{i-1}))^2 \right)^{1/2}
\]

\[
= \sum_{i=1}^{n} \left( (\dot{x}(t_i)^* \Delta t_i)^2 + (\dot{y}(t_i)^* \Delta t_i)^2 + (\dot{z}(t_i)^* \Delta t_i)^2 \right)^{1/2}
\]

Mean value theorem; \( t_i^* \in [t_{i-1}, t_i] \) and \( \Delta t := |t_i - t_{i-1}| \)

\[
= \sum_{i=1}^{n} \left( (\dot{x}(t_i^*))^2 + (\dot{y}(t_i^*))^2 + (\dot{z}(t_i^*))^2 \right)^{1/2} \Delta t_i
\]

\[
n \to \infty \Rightarrow \int_a^b \left( (\dot{x}(t)^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2 \right)^{1/2} dt
\]

\[
= \int_a^b \| \dot{\gamma}(t) \| dt
\]

- Parameter independence. Let \( \phi : [a, b] \rightarrow [a, b] \) be a diffeomorphism with \( \phi(a) = a \) and \( \phi(b) = b \). Let \( \tilde{\gamma}(s) := \gamma(\phi(s)) \). Then

\[
\text{length}(\tilde{\gamma}([a, b])) = \int_a^b \left\| \frac{d\gamma}{ds} \circ \phi(s) \right\| ds
\]

\[
= \int_a^b \left\| \phi'(s) \left\| \frac{d\gamma}{dt} \circ \phi(s) \right\| ds \right\|
\]

Let \( t = \phi(s) \) so \( dt = \phi'(s)ds \) and thus \( ds = (\phi'(s))^{-1}dt = (\phi' \circ \phi^{-1}(t))^{-1}dt \)

\[
= \int_a^b \left\| \phi' \circ \phi^{-1}(t) \left\| \frac{d\gamma(t)}{dt} \right\| \left\| \frac{dt}{\phi' \circ \phi^{-1}(t)} \right\| dt
\]

\[
= \int_a^b \left\| \frac{d\gamma(t)}{dt} \right\| dt
\]

\[
= \text{length}(\gamma([a, b]))
\]
• Example calculations — mostly no closed form for arc lengths.
  
  – First example: \( \gamma(t) = (e^t \cos(t), e^t \sin(t)) \). Then \( \dot{\gamma}(t) = e^t(\cos(t), \sin(t)) + e^t(-\sin(t), \cos(t)) \) and \( \|\dot{\gamma}(t)\| = e^t \sqrt{(\cos(t), \sin(t))^2 + (-\sin(t), \cos(t))^2)} = \sqrt{2}e^t \). Thus

  \[
  \text{length}(\gamma([0, T])) = \int_0^T \|\dot{\gamma}(t)\| dt = \sqrt{2} \int_0^T e^t dt = \sqrt{2}(e^T - 1)
  \]

  – Second example: \( \gamma(t) \) such that \( \|\dot{\gamma}\| = \text{const} \). Then

  \[
  \text{length}(\gamma([T_0, T])) = \int_{T_0}^T \|\dot{\gamma}(t)\| dt = C(T - T_0)
  \]

  Thus \( L = C(T - T_0) \) and \( T \) is almost the arc-length parameter itself. If \( C = 1 \) we say that \( \gamma \) is parametrized by arc-length.

• The arc length re-parametrization — proof that it has constant velocity. Let \( \gamma : I \to \mathbb{R} \) be a smooth curve and define the function \( \ell : I \to [0, \text{length}(\gamma(I))] \) by \( \ell(t) := \int_0^t |\dot{\gamma}(x)||dx \).

  – Note that \( \frac{d\ell(t)}{dt} = \|\dot{\gamma}(t)\| \) so that if \( \gamma \) has no points where \( \dot{\gamma} = 0 \) then \( \ell \) is invertible.

  – Define a new parameter \( s \) that satisfies \( s = \ell(t) \). So now we have \( t = \ell^{-1}(s) \) and we can define a re-parametrized version of \( \gamma \), namely \( \tilde{\gamma}(s) = \gamma(\ell^{-1}(s)) \).

  – Note that \( \|\frac{d\tilde{\gamma}(s)}{ds}\| = 1 \) because

  \[
  \frac{d\tilde{\gamma}(s)}{ds} = \frac{d\gamma}{dt} \circ \ell^{-1}(s) \frac{d\ell^{-1}(s)}{ds} = \frac{\dot{\gamma} \circ \ell^{-1}(s)}{\|\dot{\gamma} \circ \ell^{-1}(s)\|}
  \]

  – Thus \( \|\frac{d\gamma(s)}{ds}\| = 1 \) and the re-parametrized version is parametrized by arc length.

  – The arc-length parametrization is very useful theoretically (as we’ll see) but difficult to work with in practice because the arc-length can be hard to compute (i.e. finding the function \( \ell \)) and it’s inverse can then be very hard to find (i.e. inverting to find \( \ell^{-1} \)).

  – Example: we have \( s = \sqrt{2}e^t \) for the logarithmic spiral so \( t = \log(s/\sqrt{2}) \). Hence the re-parametrized version of the logarithmic spiral is

  \[
  \tilde{\gamma}(s) = \frac{s}{\sqrt{2}}(\cos(\log(s/\sqrt{2})), \sin(\log(s/\sqrt{2})))
  \]

Curvature.

• Definition of the geodesic curvature vector in an arbitrary parametrization — the normal component of the acceleration vector, normalized by the squared length of the tangent vector.

  \[
  \tilde{k}_c := \frac{1}{\|\dot{c}\|^2} \left( \ddot{c} - \frac{\langle \ddot{c}, \dot{c} \rangle}{\|\dot{c}\|^2} \dot{c} \right) = \frac{1}{\|\dot{c}\|} \left[ \frac{d}{dt} \left( \frac{\dot{c}}{\|\dot{c}\|} \right) \right]^\perp
  \]

  Rate of change of the unit tangent vector perpendicular to the curve

  • Definition of the geodesic curvature \( k_c := \|\tilde{k}_c\| \).

  • In the arc length parametrization we have \( \dot{k}_c = [\ddot{c}]^\perp \).

  • Examples: zero-acceleration curve — straight line; constant-acceleration plane curve — circle.
Frenet frame.

- Let $\gamma : \rightarrow \mathbb{R}^3$ be a curve, without loss of generality parametrized by arc-length. We will now find a canonical framing of $\gamma$, namely a choice of “moving axes” (three linearly independent vectors attached to each point $\gamma(s)$) that is best adapted to its geometry.
- Let $T(s) := \dot{\gamma}(s)$. Then $\|T(s)\| = 1$ for all $s$ since $\gamma$ is parametrized by arc-length.
- A point of non-zero curvature allows us to define a distinguished normal vector. Recall that we have $0 = \frac{d}{ds}\|\dot{\gamma}(s)\|^2 = 2\langle T(s), \ddot{T}(s) \rangle = 2\langle T(s), \ddot{k}_\gamma(s) \rangle$. Thus the curvature vector is normal to $\gamma$. Since it’s not equal to zero, we can divide by its magnitude and obtain a unit normal vector field $N(s) := \ddot{T}(s)/\|\ddot{T}(s)\|$ along $\gamma$. This is our second vector in the moving axis.
- We define the osculating plane at $\gamma(s)$ to the plane spanned by $T(s)$ and $N(s)$.
- We now define the binormal vector, the third vector in our moving axes, to be $B(s) := T(s) \times N(s)$. This is also a unit vector and is orthogonal to both $T(s)$ and $N(s)$.
- The Frenet frame for $\gamma$ is the set of moving axes $\{T(s), N(s), B(s)\}$ and is defined at each point $\gamma(s)$ where $k_\gamma(s) \neq 0$.
- The Frenet formulas explain the variation in the Frenet frame along $\gamma$. That is, we have
  \[
  \ddot{T}(s) = k_\gamma(s)N(s)
  \]
  \[
  \ddot{N}(s) = \langle \ddot{N}(s), T(s) \rangle T(s) + \langle \dot{N}(s), N(s) \rangle N(s) + \langle \dot{N}(s), B(s) \rangle B(s)
  = -k_\gamma(s)T(s) + \langle \ddot{N}(s), B(s) \rangle B(s)
  = -k_\gamma(s)T(s) - \tau_\gamma(s)B(s)
  \]
  \[
  \ddot{B}(s) = \langle \ddot{B}(s), T(s) \rangle T(s) + \langle \dot{B}(s), N(s) \rangle N(s) + \langle \dot{B}(s), B(s) \rangle B(s)
  = -\langle B(s), \ddot{T}(s) \rangle T(s) + \langle \ddot{B}(s), N(s) \rangle N(s)
  = -k_\gamma(s)\langle B(s), N(s) \rangle T(s) - \langle B(s), \dot{N}(s) \rangle N(s)
  = \tau_\gamma(s)N(s)
  \]
- Here we have introduced the torsion $\tau_\gamma(s) := -\langle \dot{N}(s), B(s) \rangle$.
- Local Theorem: Let $\gamma : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3$ be a curve with non-zero curvature. Let $k := k_\gamma(0)$ and $\tau = \tau_\gamma(0)$ and $k' = \dot{k}_\gamma(0)$. Then
  \[
  \gamma(s) \approx \gamma(0) + s\dot{\gamma}(0) + \frac{s^2}{2}\ddot{\gamma}(0) + \frac{s^3}{6}\dddot{\gamma}(0)
  = \left( s - \frac{k^2 s^3}{6} \right) T(0) + \left( \frac{s^2 k}{2} + \frac{s^3 k'}{6} \right) N(0) - \frac{k\tau s^3}{6} B(0)
  \]
  Thus locally, $k$ and $k'$ determine the amount of turning in the $\{T(0), N(0)\}$-plane, while $\tau$ and $k$ determine the amount of lifting out of the $\{T(0), N(0)\}$-plane in the $B(0)$-direction.
- Global Theorem: the Fundamental Theorem of Curves.

“Given differentiable functions $k : I \rightarrow \mathbb{R}$ with $k > 0$, and $\tau : \rightarrow \mathbb{R}$, there exists a regular curve $\gamma : I \rightarrow \mathbb{R}^3$ such that $s$ is the arc-length, $k(s)$ is the geodesic curvature, and $\tau(s)$ is the torsion. Any other curve satisfying the same conditions differs from $\gamma$ by a rigid motion.”

- A proof of the uniqueness part: differentiate $\frac{1}{2}\|\gamma(s) - \hat{\gamma}(s)\|^2$. A proof of the existence part involves solving a system of ODEs.
Bishop frame.

- The Frenet frame has an “existential” problem... I. e. it is not defined when $k_\gamma(s) = 0$. But as a paper from the 1960s asserts: *There is more than one way to frame a curve.*
- Definition. The Bishop frame gives an alternative framing of a curve.
- Variational characterization of the Bishop frame. Bending and twisting energies.
- What’s the best example?