CS 468

Differential Geometry for Computer Science

Lecture 17 — Surface Deformation
Outline

• Fundamental theorem of surface geometry.
• Some terminology: embeddings, isometries, deformations.
• Curvature flows
• Elastic deformations
The Gauss and Codazzi Equations

Recall the Gauss Equation:

\[ 0 = \langle D_Y D_X Z - D_X D_Y Z, W \rangle \]
\[ = Rm(X, Y, Z, W) + A(Y, Z)A(X, W) - A(X, Z)A(Y, W) \]

The second important equation linking intrinsic and extrinsic geometry is the Codazzi Equation.

\[ 0 = \langle D_Y D_X Z - D_X D_Y Z, N \rangle \]
\[ = \nabla A(X, Y, Z) - \nabla A(Y, X, Z) \]

These are key consistency equations which in principle completely characterize the surface.
**Fundamental Theorem of Surface Geometry**

**Theorem:**

Let $\Omega$ be an open, simply-connected subset of the plane equipped with two tensor fields $g$ and $A$ satisfying the Gauss and Codazzi equations.

Then there exists a mapping $\phi : \Omega \rightarrow \mathbb{R}^3$ of class $C^3$ such that the first and second fundamental forms of the surface $M := \phi(\Omega)$ pull back to the tensor fields $g$ and $A$.

$\phi$ is unique up to rigid motions.

**Thus:** The metric and second fundamental form determine the surface at least locally.

**And:** Changes to the surface can be characterized geometrically by how the metric and second fundamental form change.
Abstract Surfaces, Embeddings and Deformations

There is a notion of an abstract surface.

- This is a two-dimensional manifold that exists on its own, without reference to the ambient Euclidean space.

Let $M$ be an abstract surface. A map $\phi : M \rightarrow \mathbb{R}^3$ is an embedding if it is a diffeomorphism onto its image and $\phi(x) = \phi(y)$ iff $x = y$.

- This is our “usual” definition of a surface.

- Let $S = \phi(M)$. Then $M$ inherits a metric and a second fundamental form from $S$.

- Isometries are the changes of $M$ that do not change the metric.

- Isometries of $M$ may or may not involve changes of $S$.
  - Rigid motions, a spherical cap, a developable surface.

- Deformations are changes of $S$ that change both the metric and second fundamental form.
Curvature Flows

Controlled deformations of a surface arise in a number of ways.

E.g. a family $\phi_t$ of embeddings evolves by mean curvature flow if

$$\frac{d\phi_t}{dt} = H_t N_t$$

where $H_t$ is the mean curvature of $\phi_t$ and $N_t$ is their unit normal.

Note: We’ve seen this before. One can show that $\Delta \phi_t = H_t N_t$. This is Laplacian smoothing.

- So mean curvature is like heat flow except for surfaces! This wants to dissipate curvature.
- Analytical properties: short-time existence and smoothing.
- Non-linear — long-time existence in doubt, singularities
MCF of Curves in the Plane

A curve $\gamma_t : \mathbb{S}^1 \to \mathbb{R}^2$ evolves by curve shortening flow if it satisfies

$$\frac{\partial \gamma_t}{\partial t} = k_t N_t$$

where $k_t$ is the geodesic curvature of $\gamma_t$ and $N_t$ is its unit normal.

- Suppose that $\gamma_t = \gamma_t(s)$ is parametrized by arc length. By the Frenet formulas, the tangent vector satisfies $T_t := \frac{\partial \gamma_t}{\partial s}$ and

$$\frac{\partial^2 \gamma_t}{\partial s^2} = \frac{\partial T_t}{\partial s} = k_t N_t = \frac{\partial \gamma_t}{\partial t}$$

- Exact solution for a round circle — collapsing to a point.

- Some results.
  - The Gage-Hamilton theorem for convex curves (preservation of convexity and convergence to a round point in finite time).
  - The Grayson theorem for embedded curves (convergence to a round point in finite time).
Grayson’s Theorem

\[ \gamma(0) \]
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singularity
MCF of Surfaces

What changes for MCF of surfaces in $\mathbb{R}^3$?

- We still have a non-linear parabolic system.
- Exact solution for a round sphere — collapsing to a point.
- The Huisken theorem for convex surfaces (convergence to a round point in finite time).
- Singularities of the mean curvature flow in general — it’s a tricky business! E.g. a dumb-bell surface.
Elasticity theory characterizes deformations of an object by means of how they affect the induced metric in the reference object.

\[ g_{\text{original}} = \delta \]

\[ g_{\text{deformed}} = \phi_t^* \delta := D\phi_t^T D\phi_t \]

i.e. the pullback of \( \delta \) under \( \phi_t \)
Basic Principles

Let \( \rho \) be the density of \( M \) and \( \rho_t := \rho \circ \phi_t^{-1} \) be the density of \( S_t \). Also, let \( v_t := \frac{\partial \phi_t}{\partial t} \circ \phi_t^{-1} \) be the spatial velocity of points in \( S_t \).

The nonlinear equations of elasticity follow from three principles.

1. Mass is conserved:

\[
\frac{\partial \rho_t}{\partial t} + \nabla \cdot (\rho_t v_t) = 0
\]

2. Momentum is conserved: Applied body forces

\[
\frac{\partial \rho_t v_t^i}{\partial t} + \sum_j \nabla_j (\rho_t v_t^j v_t^i) = \rho_t b_t^i + \sum_j \sigma_j^i N^j
\]

Cauchy Stress Tensor
(The force per unit area on an internal surface element \( \perp N \))
Basic Principles

The metric hasn’t appeared yet. It encodes the response of the material to the applied forces.

Define the Dual Right Cauchy-Green Strain Tensor by

\[ E = \frac{1}{2} (g_{\text{deformed}} - g_{\text{orig}}) = \frac{1}{2} (D\phi_t^\top D\phi_t - \delta) \]

Now we have our third principle.

3. The constitutive relation:

\[ \sigma = \mathcal{P}(C \odot E) \]

where \( C \) is the elasticity tensor and \( \mathcal{P} \) is the Piola transform that converts quantities in \( M \) to quantities in \( S_t \).
Elastic Equilibrium

An object is in elastic equilibrium if $\phi_t$ is constant in $t$.

For hyperelastic materials we can characterize an equilibrium by means of a variational principle.  

$$\phi_{\text{equil}} = \arg \min J(\phi) := \int_M W(x, E(x)) \, dx$$

Here, $W$ is the stored energy function. It can take many forms, depending on the material properties.
Elastic Shells

Consider a thin reference object \( M := M_0 \times [-\varepsilon, \varepsilon] \) of thickness \( 2\varepsilon \).

Propose the form \( \Phi(x^1, x^2, x^3) := \phi(x^1, x^2) + x^3 N(x^1, x^2) \) for embedding \( M \) into \( \mathbb{R}^3 \), where \( \phi : M_0 \to \mathbb{R}^3 \) embeds \( M_0 \) as a surface.

The plan:

- Make several material and geometric hypotheses about \( \Phi \).
- Expand the 3D equations in \( \varepsilon \).
- Derive formal equations satisfied by \( \phi \) on \( S \) and \( M_0 \) alone.
- Prove convergence as \( \varepsilon \to 0 \).
- Tricky business!

\[ M = M_0 \times [-\varepsilon, \varepsilon] \]
Elastic Equilibrium of Shells

Equilibrium configurations of shells can also be shown to minimize an energy functional.

\[
\phi_{equil} = \arg \min_{\phi} \ k_s \int_S C(\delta g, \delta g) \, dx + k_b \int_S C(\delta A, \delta A) \, dx
\]

where \( \delta g := g_{orig} - g_{deformed} \) and \( \delta A := A_{orig} - A_{deformed} \) and \( k_s, k_b \) are constants depending on assumptions and shell thickness.

Under certain assumptions on \( C \), we can simplify to

\[
\text{stretching energy} = \int_S \|g_{orig} - g_{deformed}\|^2 \, dx
\]
\[
\text{bending energy} = \int_S \|A_{orig} - A_{deformed}\|^2 \, dx
\]