Lecture 15 — Isometries, Rigidity and Curvature
Outline

- Geodesic normal coordinates
- Local rigidity — Gauss curvature and the Theorema Egregium
- Isometries and isometry invariance
- Global rigidity — Gauss-Bonnet theorem
The Exponential Map

Recall: The geodesic exponential map of a surface $S$ at $p \in S$ is the mapping $\exp_p : T_p S \rightarrow S$ defined by

$$\exp_p(V) := \gamma(1)$$

where $\gamma$ is the unique geodesic through $p$ in direction $V$.

Key facts:

- There are open sets $U \subseteq T_p S$ containing the origin and $V \subseteq S$ containing $p$ so that $\exp_p : U \rightarrow V$ is a diffeomorphism.
- W.l.o.g. $U$ and $V$ are geodesically convex.
- The curve $t \rightarrow \exp_p(tV)$ is a geodesic for each $V \in U$. 
Geodesic Normal Coordinates

We can use $\exp_p$ to create local coordinates near $p \in S$.

- Choose an orthonormal basis $e_1, e_2$ for $T_p S$.
- Choose $r$ so that $x^1 e_1 + x^2 e_2 \in \mathcal{U}$ for all $(x^1, x^2) \in B_r(0) \subseteq \mathbb{R}^2$.
- Define $\phi : B_r(0) \to S$ by $\phi(x^1, x^2) := \exp_p(x^1 e_1 + x^2 e_2)$.

Properties:

- Straight lines through the origin in $B_r(0)$ are geodesics.
- The induced metric is Euclidean at the origin in $B_r(0)$.
- The Christoffel symbols vanish at the origin in $B_r(0)$.

$$g_{ij}(x) = \delta_{ij} + \mathcal{O}(\|x\|^2) \quad x \in B_r(0)$$
Local Rigidity

We can thus find coordinates that make the induced metric Euclidean to first order at any point.

**Question:** Can we do better?

- For instance, can we achieve the ultimate simplification — can we make the metric Euclidean in an entire neighbourhood?
- Or how about just Euclidean to second order at any point?

**NO!** A fundamental fact is

- The equations we’d have to solve to achieve a Euclidean metric to more than second order are **overdetermined**.
- There are integrability conditions that have to hold:

\[
0 = \frac{\partial \Gamma^s_{jk}}{\partial x_i} - \frac{\partial \Gamma^s_{ik}}{\partial x_j} + \Gamma^t_{jk} \Gamma^s_{it} - \Gamma^t_{ik} \Gamma^s_{jt} \quad \text{for all } i, j, k, s
\]
Gauss’ Totally Awesome Theorem

We can interpret the integrability condition in terms of curvature.

• Define the Riemann curvature (3,1)-tensor of \( S \) by

\[
Rm(X, Y, Z) := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z - \nabla_{[X,Y]} Z
\]

• Thus we can expand \( Rm = \sum_{ijks} R_{ijk}^s \omega^i \otimes \omega^j \otimes \omega^k \otimes E_s \) where

\[
R_{ijk}^s = \frac{\partial \Gamma_{jk}^s}{\partial x^i} - \frac{\partial \Gamma_{ik}^s}{\partial x^j} + \Gamma_{jk}^t \Gamma_{it}^s - \Gamma_{ik}^t \Gamma_{jt}^s
\]

• Now we have the Theorema Egregium of Gauss that relates the Riemann curvature tensor to the second fundamental form:

\[
R_{ijk}^s + (A_{jk}^s A_i^s - A_{ik}^s A_j^s) = 0 \quad \text{where} \quad A_i^s = \sum_t g^{st} A_{it}
\]
Interpretation

Let $R_{ijkl} := \sum_s g_{\ell s} R_{ijk}^s$ be the Riemann curvature $(4,0)$-tensor.

In two dimensions, the Theorema Egregium shows that the only independent term in $R_{ijkl}$ is

$$R_{1212} = -\left( A_{11} A_{22} - A_{12}^2 \right)$$

The determinant of $A$ (in an ONB) is the product of the principal curvatures, also known as the Gauss curvature!

It's an intrinsic quantity!
Isometries

**Def:** Surfaces $S$ and $S'$ with metrics $g$ and $g'$ are isometric if there exists $\phi : S \rightarrow S'$ s.t. for all $X_p, Y_p \in T_pS$ and all $p \in S$ we have

$$g'(D\phi(X_p), D\phi(Y_p)) = g(X_p, Y_p).$$

i.e. the intrinsic geometry is preserved at corresponding points.

**Examples:**

- Isometries induced from rigid motions of $\mathbb{R}^3$.
- Purely intrinsic isometries.
  - Non-planar developable surfaces.
  - Catenoid and helicoid.
  - Amphora and inverted amphora.
  - Infinitesimal isometries and Killing vector fields.
The Catenoid and the Helicoid Are Isometric
Rigidity

Isometries are rare.

**Fact:** Curvature is a local invariant under isometry.

- The key obstruction to the existence of local isometries.
- I.e. surfaces with different curvatures can’t be isometric.
- But surfaces with the same curvature are so — locally.
- Example: surfaces of constant curvature.
  - The exponential maps can be used for this purpose.
  - Choose a basis for $T_p M$ and $T_q N$.
  - Now consider $\exp_q^N \circ \left( \exp_p^M \right)^{-1}$.

Globally, it’s more complicated!
Gauss-Bonnet Theorem

The Gauss-Bonnet Theorem shows that curvature is also a global invariant with a connection to topological type.

**Theorem:** Let $S$ be a regular, oriented surface with piecewise-smooth boundary consisting of consecutive curves $C_1, \ldots, C_n$. Let $\theta_i$ be the external angle at the $C_i \rightarrow C_{i+1}$ transition. Then the Gauss-Bonnet formula holds:

$$\sum_i \int_{C_i} k_{C_i}(s) ds + \sum \theta_i + \int_S K dA = 2\pi \chi(S)$$

where $k_C$ is the geodesic curvature of $C$ and $K$ is the Gauss curvature of $S$ and $\chi(S)$ is the Euler characteristic of $S$. 
Sketch of the Proof

• Carve $S$ up into small triangular patches, each topologically equivalent to a disk.

• Apply the local Gauss-Bonnet theorem to each patch, and add up all contributions appropriately.

• The local Gauss-Bonnet theorem itself has a number of steps.
  1. Introduce an orthogonal coordinate system.
  2. Define the angle $\phi$ between vector fields $V, W$ along a curve $\gamma$.
  3. Relate $\phi'$ to the covariant derivatives of $V, W$ along $\gamma$.
  4. Let $V = \gamma'$ and $W$ be a coordinate vector field. Relate $\vec{k}_\gamma$ to $\phi'$.
  5. Integrate this relationship along $\gamma$ and apply Green’s Theorem.
  6. Apply the theorem of turning tangents.