Discrete Exterior Calculus

CS 468, Spring 2013
Differential Geometry for Computer Science

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<math_review>
Vector Calculus

\[ \text{div } \vec{v} \equiv \nabla \cdot \vec{v} \equiv \sum_i \frac{\partial v_i}{\partial x_i} \]

\[ \text{curl } \vec{v} \equiv \nabla \times \vec{v} \equiv \cdots \]

\[ \Delta f \equiv \nabla \cdot \nabla f \equiv \sum_i \frac{\partial^2 f}{\partial x_i^2} \]
Famous Theorems (in $\mathbb{R}^2$)

\[ \int \text{div} \, \vec{v} \, dA = \int_{\partial} \vec{v} \cdot \vec{n} \, dl \]

“Divergence Theorem”

\[ \int \text{curl} \, \vec{v} \, dA = \int_{\partial} \vec{v} \cdot \vec{t} \, dl \]

“Green’s Theorem”
Famous Theorems (in $\mathbb{R}^2$)

\[ \int \text{div} \, \vec{v} \, dA = \int_{\partial} \vec{v} \cdot \vec{n} \, dl \]

“Divergence Theorem”

\[ \int \text{curl} \, \vec{v} \, dA = \int_{\partial} \vec{v} \cdot \vec{t} \, dl \]

“Green’s Theorem”
Extension of vector calculus to surfaces (and manifolds).
New Rule

Everything must be intrinsic!

Vector fields are tangent!
For each point $p$ on a surface:

$k$ vectors in the tangent space at $p$ \rightarrow \text{Differential } k\text{-form} \rightarrow R$

$k$-linear
Alternating

[Sanity check: In $n$ dimensions, $p$-forms are zero for $p > n$.]
Easiest Example

\[ f : \Sigma \rightarrow \mathbb{R} \]

\text{o-form}
Vector field
$\vec{v} : \Sigma \rightarrow T\Sigma \subset \mathbb{R}^3$

1-form $\omega$:
$\omega(\vec{x}) = \vec{v} \cdot \vec{x}$

Differential One-Forms
Trivia: Musical Isomorphisms

\[ \omega^i = \sum_j g^{ij} v_j \]

Sharp operator raises indices

Elgar, Cello Concerto
Trivia: Musical Isomorphisms

Flat operator lowers indices

\[ v_i = \sum_j g_{ij} \omega^j \]
Evaluating One-Forms

\[ \omega(\vec{v}) = \sum_i \omega^i v_i \]
Zoo of Operators

\[ \omega^\# \] 1-form to vector

\[ \mathbf{v} \] Vector to 1-form

\[ d\omega \] Exterior derivative

\[ \star \omega \] Dual
\[ k \text{-forms} \to (n-k) \text{-form (plane to its normal)} \]

\[ \omega_1 \wedge \omega_2 \] Product of forms
\[ k,p \text{-forms} \to (k+p) \text{-form (cross product!)} \]
Integration of $k$-Forms

$\int_{\gamma} \omega \equiv \int_{\gamma} \omega(T) \, ds$

Measures amount of $\omega$ parallel to $\gamma$

Integrate on $k$-dimensional objects
Stokes’ Theorem

\[ \int d\omega = \int_{\partial \Omega} \omega \]
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Discrete version of exterior calculus.

\[ \omega \# \quad \nabla^b \quad \omega_1 \land \omega_2 \quad \star \omega \quad d \omega \quad \ldots \]
Recall:
Oriented Simplicial Complex
Recall: Dual Complex
Store integrals of forms!
The Trick

Discrete o-form

\[ \int \omega = f(v) \rightarrow \mathbb{R}^{|V|} \]

Store *integrated* quantities!
The Trick

Discrete 1-form

\[ \int_e \omega \rightarrow \mathbb{R}^{\vert E \vert} \]
The Trick

Discrete 2-form

Store integrated quantities!
Stokes’ Theorem

\[ \int d\omega = \int_\partial \omega \]

Implemented on homework 3!
Exterior Derivative

\[ d \in \mathbb{R}^{|E| \times |V|} \]

consists of 1, 0, -1

\[ \int_e d\omega = \int_{\partial e} \omega = \omega_2 - \omega_1 \]
Exterior Derivative

\[ d \in \mathbb{R}^{\left| F \right| \times \left| E \right|} \]

consists of 1, 0, -1

\[ \int_t \frac{d\omega}{\partial t} = \omega = \omega_1 - \omega_2 + \omega_3 \]
Exterior Derivative

$$d \in \mathbb{R}^{\mid F \mid \times \mid E \mid}$$

Haven’t made any approximations yet!

$$\int_{t} d\omega = \int_{\partial t} \omega = \omega_{1} - \omega_{2} + \omega_{3}$$
Observation

“$d^2 = 0$”

Proved on homework 3!

Two different $d$ matrices
Hodge Star: Idea

Moves to dual mesh
Hodge Star

Primal 2-form
Dual 0-form

Moves to dual mesh
Hodge Star

Primal 1-form
dual 1-form

Moves to dual mesh
Hodge Star Matrices

primal

\[
\begin{array}{c|c|c}
\hline
\text{primal} & \text{dual} & \text{primal} \\
\hline
\text{dual} & \text{primal} & \text{dual} \\
\hline
\end{array}
\]

http://brickisland.net/cs177/
Hodge Star Matrices

primal

dual

This is where approximations appear.
\[ \star_{ii} = \text{Area}(\text{triangle } i)^{-1} \]
Primal 1-Form / Dual 1-Form

Ratio of edge lengths

\[ \star \omega = \frac{|e_\star|}{|e|} \omega \]

http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf
Primal 1-Form / Dual 1-Form

\[
\frac{\|e_{ij}^*\|}{\|e_{ij}\|} = \frac{1}{2} (\cot \alpha_j + \cot \beta_j)
\]

Choice of dual: Circumcenter

http://users.cms.caltech.edu/~keenan/pdf/DGPDEC.pdf

Omit calculation

Enough already!
Primal 0-Form / Dual 2-Form

\[ \star_{ii} = \text{Area}(\text{cell } i) \]
Recall:

Barycentric Lumped Mass

http://www.alecjacobson.com/weblog/?p=1146

Area/3 to each vertex
Additional Options

Barycentric cell

\[ c_i = \text{barycenter of triangle} \]

Voronoi cell

\[ c_i = \text{circumcenter of triangle} \]

Mixed cell
Mixed Voronoi Cell

If $\theta < \pi/2$, $c_i$ is the circumcenter of the triangle $(v_i, v, v_{i+1})$

If $\theta \geq \pi/2$, $c_i$ is the midpoint of the edge $(v_i, v_{i+1})$

$$A(v) = \sum_{v_i \in N(v)} \left( \text{Area}(c_i, v, (v + v_i)/2) + \text{Area}(c_{i+1}, v, (v + v_i)/2) \right)$$
Discrete deRham Complex

0-forms (vertices) \[ \delta \rightarrow d \]

1-forms (edges) \[ \delta \rightarrow d \]

2-forms (faces) \[ \delta \rightarrow d \]

3-forms (tets) \[ \delta \rightarrow d \]
Co-Differential

**Theorem.** \( \langle d\beta, \alpha \rangle = -\langle \beta, \star d \star \alpha \rangle \)

\[ \delta \equiv - \star d \star \]
Hodge Laplacian

\[ \Delta = d \star d \star + \star d \star \star d \]
$\Delta = d \star d \star + \star d \star d$

Cotangent Laplacian
o-Form Laplacian

\[ \Delta = d \star d \star + \star d \star d \]

Area weights
Helmholtz-Hodge Decomposition

Divergence free

Curl free

Harmonic
\[ \omega = \delta \beta + d \alpha + \gamma \]

where \( d \gamma = 0 \), \( \delta \gamma = 0 \)
Computing the Decomposition

\[ \omega = \delta \beta + d\alpha + \gamma \]

where \( d\gamma = 0, \delta \gamma = 0 \)

\[ \delta d\alpha = \delta \omega \]

\[ dl\delta \beta = d \omega \]

\[ \gamma = \omega - \delta \beta - d\alpha \]
One-Form Laplacian Eigenforms

\[ \omega = \delta \beta + d\alpha + \gamma \]
where \( d\gamma = 0, \delta \gamma = 0 \)

\[
\lambda(-\ast d\bar{\beta} + d\alpha + \gamma) = \lambda \omega = \Delta \omega
\]

\[
= (d \ast d \ast + \ast d \ast d)(\delta \beta + d\alpha + \gamma)
\]

\[
= (d \ast d \ast + \ast d \ast d)(-\ast d \ast \beta + d\alpha)
\]

\[
= -\ast d \ast d \ast d \ast \beta + d \ast d \ast d \ast d\alpha
\]

\[
= -\ast d \Delta \bar{\beta} + d \Delta \alpha
\]

Conclusion: For \( \lambda \neq 0 \), they’re obtained by \( d \) and \( \ast d \) of Laplacian eigenfunctions.
The Helmholtz-Hodge Decomposition - A Survey

Harsh Bhatia, Student Member IEEE, Gregory Norgard, Valerio Pascucci, Member IEEE, and Peer-Timo Bremer, Member IEEE

Abstract—The Helmholtz-Hodge Decomposition (HHD) describes the decomposition of a flow field into its divergence-free and curl-free components. Many researchers in various communities like weather modeling, oceanology, geophysics and computer graphics are interested in understanding the properties of flow representing physical phenomena such as incompressibility and vorticity. The HHD has proven to be an important tool in the analysis of fluids, making it one of the fundamental theorems in fluid dynamics. The recent advances in the area of flow analysis have led to the application of the HHD in a number of research communities such as flow visualization, topological analysis, imaging, and robotics. However, since the initial body of work, primarily in the physics communities, research on the topic has become fragmented with different communities working largely in isolation often repeating and sometimes contradicting each others results. Additionally, different nomenclature has evolved which further obscures the fundamental connections between fields making the transfer of knowledge difficult. This survey attempts to address these problems by collecting a comprehensive list of relevant references and examining them using a common terminology. A particular focus is the discussion of boundary conditions when computing the HHD. The goal is to promote further research in the field by creating a common repository of techniques to compute the HHD as well as a large collection of example applications in a broad range of areas.

Index Terms—Vector fields, Incompressibility, Boundary Conditions, Helmholtz-Hodge decomposition.
Recommended Reading

Today will take a few random samples
Fig. 2. Sequence of images from the Hurricane Luis sequence, with eye segmented.

Fig. 1. (a) Motion field in a anticlockwise rotating hurricane sequence extracted using the BMA. (b) The divergence free potential function with a distinct maximum and corresponding contours.

Fluid Simulation

Stam. “Stable Fluids.” SIGGRAPH 1999. (and many others)

Incompressible: No divergence
Vector Field Editing

Separate turbulence from acoustics in solar simulation

Reconstruct VF from Noisy Samples

\[ \Phi_{df}(x) = H\phi(x) - tr \{ H\phi(x) \} I \]
\[ \Phi_{cf}(x) = -H\phi(x) \]

Macedo and Castro.
Wrapping Up for Today

- Another cotangent Laplacian
- Helmholtz-Hodge Decomposition

Many more applications!