Outline

- Linear and multilinear algebra with an inner product
- Tensor bundles over a surface
- Symmetric and alternating tensors
- Exterior calculus
- Stokes’ Theorem
- Hodge Theorem
Inner Product Spaces

Let $\mathcal{V}$ be a vector space of dimension $n$.

**Def:** An inner product on $\mathcal{V}$ is a bilinear, symmetric, positive definite function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$.

We have all the familiar constructions:

- The **norm** of a vector is $\|v\| := \sqrt{\langle v, v \rangle}$.
- Vectors $v, w$ are **orthogonal** if $\langle v, w \rangle = 0$.
- If $S$ is a subspace of $\mathcal{V}$ then every vector $v \in \mathcal{V}$ can be uniquely decomposed as $v := v^\parallel + v^\perp$ where $v^\parallel \in S$ and $v^\perp \perp S$.
- The mapping $v \mapsto v^\parallel$ is the **orthogonal projection** onto $S$. 
Dual Vectors

**Def:** Let $\mathcal{V}$ be vector space. The dual space is

$$\mathcal{V}^* := \{ \xi : \mathcal{V} \to \mathbb{R} : \xi \text{ is linear} \}$$

**Proposition:** $\mathcal{V}^*$ is a vector space of dimension $n$.

**Proof:** If $\{E_i\}$ is a basis for $\mathcal{V}$ then $\{\omega^i\}$ is a basis for $\mathcal{V}^*$ where

$$\omega^i(E_s) = \begin{cases} 1 & i = s \\ 0 & \text{otherwise} \end{cases}$$
The Dual Space of an Inner Product Space

Let \( \mathcal{V} \) be vector space with inner product \( \langle \cdot, \cdot \rangle \). The following additional constructions are available to us.

- If \( v \in \mathcal{V} \) then \( v^b \in \mathcal{V}^* \) where \( v^b(w) := \langle v, w \rangle \) \( \forall w \in \mathcal{V} \).

- If \( \xi \in \mathcal{V}^* \) then \( \exists \xi^\# \in \mathcal{V} \) so that \( \xi(w) = \langle \xi^\#, w \rangle \) \( \forall w \in \mathcal{V} \).

- These are inverse operations: \( (v^b)^\# = v \) and \( (\xi^\#)^b = \xi \).

- \( \mathcal{V}^* \) carries the inner product \( \langle \xi, \zeta \rangle_{\mathcal{V}^*} := \langle \xi^\#, \zeta^\# \rangle \) \( \forall \xi, \zeta \in \mathcal{V}^* \).
Basis Representations

Let \( \{ E_i \} \) denote a basis for \( V \) and put \( g_{ij} := \langle E_i, E_j \rangle \).

**Def:** Let \( g^{ij} \) be the components of the inverse of the matrix \( [g_{ij}] \).

Then:

- The dual basis is \( \omega^i := \sum_j g^{ij} E_j \).
- If \( v = \sum_i v^i E_i \) then \( v^b = \sum_i v_i \omega^i \) where \( v_i := \sum_j g_{ij} v^j \).
- If \( \xi = \sum_i f_i \eta^i \) then \( f^\# = \sum_i f^i E_i \) where \( f^i := \sum_j g^{ij} f_j \).
- If \( \xi = \sum_i a_i \omega^i \) and \( \zeta = \sum_i b_i \omega^i \) then \( \langle \xi, \zeta \rangle = \sum_{ij} g^{ij} a_i b_j \).

**Note:** If \( \{ E_i \} \) is orthonormal then \( g^{ij} = \delta_{ij} \) and \( v_i = v^i \) and \( \xi^i = \xi_i \).
Tensors

Let $\mathcal{V}$ be a vector space of dimension $n$.

Tensors are “multilinear functions on $\mathcal{V}$ with multi-vector output.”

**Def:** The space of $k$-covariant and $\ell$-contravariant tensors is

$$\mathcal{V}^* \otimes \cdots \otimes \mathcal{V}^* \otimes \mathcal{V} \otimes \cdots \otimes \mathcal{V} := \left\{ \begin{array}{l} f : \mathcal{V} \times \cdots \times \mathcal{V} \rightarrow \mathcal{V} \times \cdots \times \mathcal{V} \\ \text{such that } f \text{ is multilinear} \end{array} \right\}$$

**Basic facts:**

- Vector space of dimension $n^{k+\ell}$. Basis in terms of $E_i$’s and $\omega^i$’s.
- Inherits an inner product from $\mathcal{V}$ and has $\sharp$ and $\flat$ operators.
- There are contractions (killing a $\mathcal{V}$ factor with a $\mathcal{V}^*$ factor).
Symmetric Bilinear Tensors

A symmetric \((2,0)\)-tensor is an element \(A \in \mathcal{V}^* \otimes \mathcal{V}^*\) such that \(A(v, w) = A(w, v)\) for all \(v, w \in \mathcal{V}\).

Some properties:

- In a basis we have \(A = \sum_{ij} A_{ij} \omega^i \otimes \omega^j\) with \(A_{ij} = A_{ji}\).

- We define an associated self-adjoint \((1,1)\)-tensor \(S \in \mathcal{V}^* \otimes \mathcal{V}\) with the formula \(A(v, w) := \langle S(v), w \rangle\).

- In a basis we have \(S = \sum_{ij} S^j_i \omega^i \otimes E_j\) where \(S^j_i = \sum_k g^{kj} A_{ik}\).

- If \(v = \sum_i v^i E_i\) and \(w = \sum_i w^i E_i\) then \(\langle v, w \rangle = [v]^\top [g][w]\) and \(A(v, w) = [v]^\top [A][w]\) and \(S = [g]^{-1}[A]\).

- The contraction of \(A\) equals the trace of \(S\) equals \(\sum_{ij} g^{ij} A_{ij}\).

Example:

\(A = 2^{nd}\) FF and \(S = \text{shape operator.}\)
Alternating Tensors

A $k$-form is an element $\sigma \in \mathcal{V}^* \otimes \cdots \otimes \mathcal{V}^*$ such that for all $v, w \in \mathcal{V}$ and pairs of slots in $\sigma$ we have

$$\sigma(\ldots v \ldots w \ldots) = -\sigma(\ldots w \ldots v \ldots)$$

“Alternating $(k, 0)$-tensor”

**Fact:** If dim $\mathcal{V} = 2$ then only $k = 0, 1, 2$ are non-trivial.

$\text{Alt}^0(\mathcal{V}) = \mathbb{R}$ and $\text{Alt}^1(\mathcal{V}) = \mathcal{V}^*$ and $\text{Alt}^2(\mathcal{V}) \cong \mathbb{R}$

**Duality:** if $\mathcal{V}$ has an inner product

- The area form $dA \in \text{Alt}^2(\mathcal{V})$

  $$dA(v, w) := \left[ \begin{array}{c} \text{Signed area of} \\ \text{parallelogon } v \wedge w \end{array} \right]$$

- The Hodge-star operator $*$

  $$\omega \wedge *\tau := \langle \omega, \tau \rangle \ dA$$

**Basis:** The element $\omega^1 \wedge \omega^2$

Let $v = \sum_i v^i E_i$ and $w = \sum_i w^i E_i$. Then we define it via

$$\omega^1 \wedge \omega^2(v, w) := \det([v \ w])$$

$*dA = 1$ and if $\omega \in \text{Alt}^1(\mathcal{V})$ then

$*1 = dA$ \qquad *\omega(v) = \omega(R_{\pi/2}(v))$
Tensor Bundles on a Surface

Let $S$ be a surface and let $\mathcal{V}_p := T_p S$.

**Def:** The bundle of $(k, \ell)$-tensors over $S$ attached the vector space $\mathcal{V}_p^{(k, \ell)} := \mathcal{V}_p^* \otimes \cdots \otimes \mathcal{V}_p^* \otimes \mathcal{V}_p \otimes \cdots \otimes \mathcal{V}_p$ at each $p \in S$.

**Def:** A section of this bundle is the assignment $p \mapsto \sigma_p \in \mathcal{V}_p^{(k, \ell)}$.

Examples:

- $k = \ell = 0$ — sections are functions on $S$
- $k = 0, \ell = 1$ — sections are vector fields on $S$
- $k = 1, \ell = 0$ — sections are one-forms on $S$
- $k = 2, \ell = 0$ and symmetric — sections are a symmetric bilinear form at each point. E.g. the metric and the $2^{nd}$ FF.
- $k = 2, \ell = 0$ and antisymmetric — sections are two-forms on $S$. E.g. the area form.
The covariant derivative extends naturally to tensor bundles.

**A formula:** Choose a basis and suppose

\[
\sigma := \sum_{ijkl} \sigma_{ij}^{k \ell} \omega^i \otimes \omega^j \otimes E_k \otimes E_\ell
\]

is a tensor. Then

\[
\nabla \sigma := \sum_{ijkls} \nabla_s \sigma_{ij}^{k \ell} \left[ \omega^i \otimes \omega^j \otimes E_k \otimes E_\ell \right] \otimes \omega_s
\]

is also a tensor, where

\[
\nabla_s \sigma_{ij}^{k \ell} := \frac{\partial \sigma_{ij}^{k \ell}}{\partial x^s} - \Gamma^{t}_{is} \sigma_{tj}^{k \ell} - \Gamma^{t}_{js} \sigma_{it}^{k \ell} + \Gamma^{i}_{ts} \sigma_{lj}^{k \ell} + \Gamma^{\ell}_{ts} \sigma_{ij}^{kt}
\]
Exterior Differentiation

**Def:** The exterior derivative is the operator \( d : \text{Alt}^k(S) \rightarrow \text{Alt}^{k+1}(S) \) defined as follows.

- Choose a basis.
- If \( f \in \text{Alt}^0(S) \) we define \( df \) geometrically by \( df(V) := V(f) \) or
  \[
  df = \sum_i \frac{\partial f}{\partial x^i} \omega^i \quad \text{Thus} \ (df)^\sharp = \nabla f
  \]
- If \( \omega = \sum_i a_i \omega^i \in \text{Alt}^1(S) \) then \( d\omega = \left( \frac{\partial a^1}{\partial x^2} - \frac{\partial a^2}{\partial x^1} \right) \omega^1 \wedge \omega^2 \)
- If \( \omega = a \omega^1 \wedge \omega^2 \in \text{Alt}^0(S) \) then \( d\omega = 0 \).

**Basic Facts:**

- \( dd\omega = 0 \) for all \( \omega \in \text{Alt}^k(S) \) and all \( k \).
- \( d\omega = \text{Antisym}(\nabla\omega) \).
The Co-differential Operator

**Def:** The co-differential is the $L^2$-adjoint of $d$. It is therefore an operator $\delta : \text{Alt}^{k+1}(S) \rightarrow \text{Alt}^k(S)$ that satisfies

$$\int_S \langle d\omega, \tau \rangle \, dA = \int_S \langle \omega, \delta\tau \rangle \, dA$$

It is given by $\delta := - * d *$.

**Interpretations:**

- If $f$ is a function, then $(df)\sharp = \nabla f$.
- If $X$ is a vector field, then $\delta X^b = \text{div}(X)$.
- If $X$ is a vector field, then $dX^b = \text{curl}(X) \, dA$.
- If $f$ is a function, then $(\delta(f \, dA))^b = R_{\pi/2}(\nabla f)$.
Stokes’ Theorem

Intuition: Generalization of the Fundamental Theorem of Calculus.

Suppose that $c$ be a $(k + 1)$-dimensional submanifold of $S$ with $k$-dimensional boundary $\partial c$. Let $\omega$ be a $k$-form on $S$. Then:

$$\int_c d\omega = \int_{\partial c} \omega$$

Interpretations:

- The divergence theorem:

$$\int_S \text{div}(X) dA = \int_{\partial S} \langle N_{\partial S}, X \rangle d\ell$$

- Etc.
The Hodge Theorem

**Theorem:** $\text{Alt}^1(S) = d\text{Alt}^0(S) \oplus \delta\text{Alt}^2(S) \oplus \mathcal{H}^1$ where $\mathcal{H}^1$ is the set of harmonic one-forms:

\[
h \in \mathcal{H}^1 \iff dh = 0 \text{ and } \delta h = 0
\]

\[
\iff (d\delta + \delta d) h = 0
\]

"Hodge Laplacian"

**Corollary:** Every vector field $X$ on $S$ can be decomposed into a "gradient" part, a "divergence-free" part, and a "harmonic part."

\[
X = \nabla \phi + \text{curl}(\nabla \psi) + h^\# \quad \text{with } h \in \mathcal{H}^1
\]

Another deep mathematical result:

**Theorem:** $\dim(\mathcal{H}^1) = 2\chi(S)$. This is a topological invariant.