Derivatives in Euclidean space.

- Derivatives of a scalar function in the direction of a vector.
- Derivatives of vector fields viewed as \( n \)-tuples of scalar functions along curves:
  \[
  [\nabla_X Y]_p := \sum_i \lim_{\varepsilon \to 0} \varepsilon^{-1}(Y_i(p + \varepsilon X) - Y_i(p))E_i.
  \]
- Four properties of the vector field derivative: the \( C^\infty \)-linearity in the \( X \)-slot, the \( \mathbb{R} \)-linearity in the \( Y \)-slot, the Leibniz rule in the \( Y \)-slot, the metric compatibility property.
- Derivatives of vector fields when moving frames are involved. Christoffel symbols.

Derivatives of objects defined on a surface.

- Derivatives of a scalar function in the direction of a vector on a surface.
- Derivatives of a vector field: there are problems...
  - The geometric definition of differentiation (limit of finite differences) fails since the derivative of a vector field on the surface may not be tangent to the surface. What’s the alternative — how can we compare vectors in different tangent spaces?
  - Using a parametrization involves moving frames (for a parametrization \( \phi : \Omega \to \mathbb{R}^3 \) the moving frame is \( E_i := D\phi(\partial/\partial u^i) \)). But still derivatives may not be tangent to the surface.
- Definition of the covariant derivative as \( \nabla_X Y := [\nabla_X Y] \parallel \). Given vector fields \( X, Y \) then \( (\nabla_X Y)_p \) depends on \( X_p \) and \( Y \) along a curve passing tangentially through \( X_p \) only. Alternate notation \( \frac{DY}{dt} \) for \( \nabla_X Y \) when \( c : I \to \mathbb{R} \) is a curve and \( X := \dot{c} \) and \( Y \) is defined along \( c \).
- Thus we have a relationship to second fundamental form \( \nabla_X Y = A(X,Y)N + \nabla_X Y \).
- The four properties of the vector field derivative revisited.
- The Lie bracket of two vector fields: the Lie bracket of two coordinate vector fields vanishes; therefore it’s a tangent vector field if the individual vector fields are. The torsion-free property.

Parallel transport.

- Definition of parallel transport and its properties. The tangent vector of a geodesic is parallel transported along itself. Geodesic equations.
- A new geometric definition of differentiation — comparison of vectors in different tangent spaces using parallel transport.

The fundamental lemma of Riemannian geometry.

- The Christoffel symbols
- Fundamental Lemma of Riemannian Geometry. Covariant derivatives are intrinsic.

Gradient, divergence and Laplacian.

- The gradient vector.
- Definition of divergence using an orthonormal frame. Independence of frame.
- The divergence theorem.
- The Laplacian. Harmonic functions, etc.. Integration-by-parts formulas.