Problem 1 (30 points).

(a) We defined the coefficients of the Riemann curvature \((3,1)\)-tensor with respect to the coordinate basis \(E_1, E_2\) by

\[ \sum_s R_{ijk}^s E_s := \nabla_{E_j} \nabla_{E_i} E_k - \nabla_{E_i} \nabla_{E_j} E_k. \]

Derive the formula

\[ R_{ijk}^s = \frac{\partial \Gamma_{jk}^s}{\partial x^i} - \frac{\partial \Gamma_{ik}^s}{\partial x^j} + \sum_t \Gamma_{jk}^t \Gamma_{it}^s - \sum_t \Gamma_{ik}^t \Gamma_{jt}^s. \]

(b) We defined the coefficients of the Riemann curvature \((4,0)\)-tensor by \(R_{ijk}^\ell := \sum_s g_{\ell s} R_{ijk}^s\) or equivalently by \(R_{ijk}^\ell := g(\nabla_{E_j} \nabla_{E_i} E_k - \nabla_{E_i} \nabla_{E_j} E_k, E_\ell)\). Use Gauss’ Theorema Egregium to verify the so-called symmetries of the curvature tensor:

\[ R_{ijk}^\ell = -R_{jik}^\ell \quad R_{ijk}^\ell = -R_{ijk}^\ell \quad R_{ijk}^\ell = R_{kij}^\ell. \]

(c) Show that on a 2-dimensional surface, the only independent component of the Riemann curvature \((4,0)\)-tensor is \(R_{1212}\). In other words, show that all other components of \(R_m\) are either zero or a multiple of \(R_{1212}\).

(d) Use intrinsic calculations to find the Riemann curvature \((4,0)\)-tensor of the sphere. (Hints: you get to choose the parametrization of the sphere — so choose wisely; also you only need to compute \(R_{1212}\)!) (e) Find the Gauss curvature of the sphere via the second fundamental form. Compare with part (d) and verify Gauss’ Theorema Egregium.

Problem 2 (20 points). Differential geometry is all about finding good local coordinate systems for a surface \(S\) which then help prove theorems. For instance, the Gauss-Bonnet theorem uses an orthogonal parametrization. This is a parametrization \(\phi : U \subseteq \mathbb{R}^2 \to S\) with the property that \(g_{12}(x) = 0\) for all \(x \in U\). In other words, if \(E_i := \frac{\partial \phi}{\partial x^i}\) then \(\langle E_1, E_2 \rangle = 0\) at all points on \(S\) in the image of \(\phi\). (In this coordinate system, it is not necessarily the case that \(\langle E_1, E_1 \rangle = \langle E_2, E_2 \rangle = 1\). In fact, if this were to hold, then \(S\) would have a neighbourhood that is isometric to Euclidean space, which can happen if and only if the Riemann curvature tensor of \(S\) is zero inside \(U\).)

Suppose that \(\gamma : [-1, 1] \to S\) is a geodesic segment in \(S\). For every \(s \in [0,1]\), let \(N(s)\) be the unit vector in \(T_{\gamma(s)}S\) that is orthogonal to \(\gamma'(s)\). In this problem, you will prove that the mapping \(\phi(s,t) := \exp_{\gamma(s)}(tN(s))\) for \(s \in (−1,1)\) and small \(t\) is an orthogonal parametrization of a neighbourhood of \(\gamma(0)\). In fact, you will do slightly better and show that \(g_{12} = 0\) and \(g_{22} = 1\) for all \((s,t)\) in the parameter domain.
(a) Draw an informative picture. What could go wrong if \( t \) is allowed to become too large?

(b) Let \( E_1 := \frac{\partial \phi}{\partial s} \) and \( E_2 := \frac{\partial \phi}{\partial t} \). Show that \( \nabla_{E_2} E_2 = 0 \) for all \( s, t \).

(c) Show that \( \| E_2 \| = 1 \) for all \((s, t)\). (Hint: why is this true when \( s \) is arbitrary and \( t = 0 \)? Now hold \( s \) fixed and show that \( \frac{\partial}{\partial t} \| E_2 \|^2 = 0 \) for all \( t \).) Conclude that \( g_{22} = 1 \) for all \((s, t)\).

(d) Show that \( \langle E_1, E_2 \rangle = 0 \) for all \((s, t)\). (Hint: why is this true when \( s \) is arbitrary and \( t = 0 \)? Now hold \( s \) fixed and show that \( \frac{\partial}{\partial t} \langle E_1, E_2 \rangle = 0 \) for all \( t \).) Conclude that \( g_{12} = 0 \) for all \((s, t)\).

**Problem 3** (20 points). The divergence theorem says that for any smooth vector field \( X \) on a surface \( S \) with boundary \( \partial S \), we have

\[
\int_S \text{div}(X) \, dA = \int_{\partial S} \langle X, N \rangle \, ds.
\]

where \( dA \) is the Riemannian area form, \( N \) is a unit vector tangent to \( S \) but normal to \( \partial S \), and we must use an arc-length parametrization for \( \partial S \) for this equation to hold. Stokes' Theorem says that for any differential \( k \)-form \( \omega \) and \((k + 1)\)-dimensional submanifold \( c \subseteq S \) we have

\[
\int_c \omega = \int_{\partial c} d \omega.
\]

In this problem, you will show that Stokes' Theorem implies the divergence theorem for a well-chosen \( \omega \). This is a straightforward problem that the unfamiliar notation of differential forms and sharp/flat/star operators may make quite difficult. Do your best!

(a) Show that \( \text{div}(X) = - * d * \langle X', \rangle \). Hint: you need to show this at an arbitrary point \( p \in S \) using your favourite coordinate system. So work in geodesic normal coordinates centered at \( p \).

(b) Explain why \( \text{div}(X) \, dA \) can be put in the form \( d \omega \) for some form \( \omega \), and what is \( \omega \)?

(c) Apply Stokes' Theorem to \( d \omega \) and \( S \) itself. We thus get \( \int_S \text{div}(X) \, dA = \int_{\partial S} \omega \). To develop the right hand side further, you must know how to evaluate the “line integral” \( \int_{\partial S} \omega \). Suppose that we can parametrize the boundary \( \partial S \) by arc-length as a curve \( \gamma : [0, \ell] \to S \) with tangent vector \( T(s) := \gamma'(s) \). Now \( \int_{\partial S} \omega \) is defined to be \( \int_0^\ell \omega(T(s)) \, ds \). Show that \( \omega(T) = \langle X, N \rangle \) where \( N \) is the vector obtained by rotating \( T \) counterclockwise by \( \pi/2 \).

**Problem 4** (15 points). Recall that the Helmholtz-Hodge decomposition of a one-form \( \omega \) is given by \( \omega = \delta \beta + d \alpha + \gamma \), where \( d \gamma = 0 \) and \( \delta \gamma = 0 \). In lecture we argued that you can find the Helmholtz-Hodge decomposition in DEC by solving \( \delta d \alpha = \delta \omega \) and \( d \delta \beta = d \omega \) (and taking \( \gamma = \omega - \delta \beta - d \alpha \)).

(a) Argue that the operators \( \delta d \) and \( d \delta \) have null spaces for closed triangulated surfaces. Why isn’t this a hole in our technique?

(b) Compute \texttt{helmholtzHodge.m} implementing this technique and visualize the results using \texttt{problem4.m}. Notice that we have kindly provided \texttt{discreteExteriorCalculus.m} implementing the DEC matrices you will need.

**Problem 5** (15 points). As promised, we return to the problem of geodesic computation:
(a) When does the planar front approximation made in the fast marching algorithm behave well? When does it behave poorly?

(b) In 2002, Novotni and Klein proposed using a circular wavefront rather than a planar wavefront in fast marching. For the most part, the algorithm remains the same, since it is a simple extension of Dijkstra’s algorithm for shortest paths, but the update step must be changed. Without loss of generality, we’ll embed three vertices of a triangle being updated onto the plane at positions $v_1 \equiv 0$, $v_2 \equiv (v_{2x}, 0)$, and $v_3 = (v_{3x}, v_{3y})$ with $v_{3y} \geq 0$ (make sure you understand why such an embedding is possible); we know distances $d_1$ and $d_2$ but want to find $d_3$.

(i) Given $d_1$ and $d_2$, write and solve a system of equations for finding the source point $(x, y)$ of the circular wavefront.

(ii) Your system from (i) should be quadratic and thus yields two solutions. Provide a rule for choosing one of the two roots to give a single point $(x, y)$, and give an expression for $d_3$. 