Problem 1 (15 points). Let $\gamma : I \rightarrow \mathbb{R}^3$ be a smooth regular curve parametrized by arc-length and suppose $N(t)$ is a smooth choice of a unit-length vector at $\gamma(t)$ that is orthogonal to $\gamma'(t)$. Let $f : I \rightarrow \mathbb{R}$ be a smooth function. The tubular surface around $\gamma$ with radial function $f$ is defined by the parametrization

$$\phi(\theta, t) := \gamma(t) + f(t) \cos(\theta)N(t) + f(t) \sin(\theta)\gamma'(t) \times N(t).$$

(a) Draw a picture of this set-up.

(b) A torus is the surface of revolution obtained by rotating around the z-axis a vertical circle of radius $r$ whose center is located a distance $R$ from the z-axis. Parametrize this torus in the form of part (a).

(c) Find a basis for the tangent space of the torus, as well as expressions for its outward-pointing unit normal vector and matrix of the second fundamental form with respect to this basis and normal vector.

Problem 2 (15 points). Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice-differentiable function and consider its graph $S := \{(x, y, f(x, y)) : x, y \in \mathbb{R}^2\}$. Find expressions for a basis for the tangent space of $S$, the unit upward-pointing normal vector of $S$ and the matrix of the second fundamental form with respect to this basis and normal vector.

Problem 3 (30 points). In differential geometry we have the ability to construct special parametrizations that are well-adapted to a given geometric setting. Often, this is the key step in the proof of a theoretical result. In this problem, you’ll show that a regular surface is locally the graph over its tangent plane.

What this means is that we can find $V \subseteq \mathbb{R}^2$ containing the origin and a function $f : V \rightarrow \mathbb{R}$ so that the graph $\{(v^1, v^2, f(v^1, v^2)) : (v^1, v^2) \in V\}$ represents the surface $S$ in the following sense: all points $x$ near $x_0$ can be parametrized in the form $x(v^1, v^2) := x_0 + E_1 v^1 + E_2 v^2 + f(v^1, v^2)N$ where $E_1, E_2$ is
a basis for $T_{x_0}S$ and $N$ is normal to $T_{x_0}S$ with $\|N\| = 1$. Furthermore we require that $f(0,0) = 0$ and $\frac{\partial f(0,0)}{\partial x^1} = \frac{\partial f(0,0)}{\partial x^2} = 0$, the consequence of which is that $x(0,0) = x_0$ and $\frac{\partial x(0,0)}{\partial x^1} = E_1$ and $\frac{\partial x(0,0)}{\partial x^2} = E_2$.

The essence of the proof below is to find a form for the function $f$ in terms of the only thing you’re allowed to assume about the surface $S$ near $x_0$, namely the existence of a parametrization for a neighbourhood of $x_0$. This is a hard problem, but is an archetype for many differential geometric constructions. If you can master this problem, then you are well on your way to being an expert!

(a) We’ll start by writing $x - x_0 = p(x) + a(x)N$ where $p(x) \perp N$. Find $a(x)$ in terms of $x, x_0$ and $N$.

(b) Let $E_1, E_2$ be any basis for $T_{x_0}S$. Find a formula for $p(x)$ in terms of $x_0, E_1, E_2$ that looks like $p(x) := p^1(x)E_1 + p^2(x)E_2$. So in other words, you have to find formulas for $p^1(x)$ and $p^2(x)$. There will be a matrix involved that you’ll have to invert — explain why it’s invertible.

(c) If we now let $v^1 := p^1(x)$ and $v^2 := p^2(x)$ and somehow we manage to invert these equations to find $x = q(u)$ for some function $q$, we now have an expression of the form $x - x_0 = v^1E_1 + v^2E_2 + a(q(v))N$. This is exactly what we want, with $f(v) := a(q(v))$. Why can’t we do this?

(d) Here’s a way around the impasse of part (c). Introduce a parametrization for a neighbourhood of $x_0$, i.e. let $\phi: U \rightarrow \mathbb{R}^3$ parametrize $S$ near $x_0$, with $\phi(0) = x_0$ without loss of generality. Let $E_1 := \frac{\partial \phi(0,0)}{\partial u^1}$ and $E_2 := \frac{\partial \phi(0,0)}{\partial u^2}$. Let $P: V \rightarrow \mathbb{R}^2$ be the function $P(u^1, u^2) := (p^1(\phi(u^1, u^2)), p^2(\phi(u^1, u^2)))$. Now we have a relationship $v = P(u)$. Find the derivative $DP_u$ at $u = 0$ and show it’s invertible.

(e) Quote the Inverse Function Theorem correctly and argue that we have an inverse function $P^{-1}: V \rightarrow U'$ defined on some sets $V$ and $U' \subseteq U$. We can now complete part (c) rigorously — what is the final form for $f: V \rightarrow \mathbb{R}$?

(f) Show that $f(0,0) = 0$ and $\frac{\partial f(0,0)}{\partial v^0} = \frac{\partial f(0,0)}{\partial v^0} = 0$. You’ll need the formula for $DP^{-1}$ in terms of $DP$ provided by the Inverse Function Theorem at some point in this calculation.

**Problem 4** (20 points). Many geometric operators assume that a mesh is oriented, meaning that the underlying surface is orientable and that the ordering of the vertices (clockwise vs. counterclockwise) in each triangle is consistent. Complete assignCoherentOrientation.m to take a list of triangles with arbitrary orientation and generate a new list where the triangles are consistently oriented. Note that globally there are two acceptable orientations (inward normal or outward normal); leave the orientation of the first triangle the same. The script problem4.m provides some code for testing your method.

**Problem 5** (20 points). In class, we discussed “boundary operators” $d$ for an oriented triangle mesh with vertices $V$, edges $E$, and triangles $T$. While finding the boundary of a simplex takes you down one dimension (from triangles to edges to vertices), we can define an operator $d$ that does the opposite. In particular, $d$ will take one value per k-simplex and return one value per $(k + 1)$-simplex (a triangle is a 2-simplex, an edge is a 1-simplex, and a vertex is a 0-simplex). We will represent $d$ using two matrices:

- $d_{0\rightarrow 1} \in \mathbb{R}^{E \times |V|}$ will take one value per vertex and return one value per edge. You must find a list of edges $E$ and assign each an orientation (unlike the halfedge structure, we won’t double edges; just assign each a single arbitrary orientation). The row of this matrix corresponding to edge $e = v_1 \rightarrow v_2$ takes the difference of the values $f(v_2) - f(v_1)$.
• $d_{1\to2} \in \mathbb{R}^{\left|\mathcal{T}\right| \times \left|\mathcal{E}\right|}$ combines edge values to triangles with $+1$ when the edge’s orientation coincides with that of the triangle and $-1$ otherwise.

We’ll proceed in two parts:

(a) Complete boundaryOperators.m to find the matrices of these two operators. To receive credit, these operators must be sparse matrices, meaning that most of their elements are zero. Check out Matlab’s sparse method for how to construct a sparse matrix. The script problem5.m provides some testing material. As a challenge, try to write this method without using any loops, which are slow in Matlab.

(b) Show that $d_{1\to2}d_{0\to1} \equiv 0$. 