Topology Preserving Edge Contractions: What are they and how do we find them?

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Jon McAllister
The contraction of an edge $ab$ is a local transformation of $K$ that replaces $St\, ab = St\, a \cup St\, b$ by the star of a new vertex, $St\, c$.

What can go wrong?
Basic definitions

• The closure of \( B \subseteq K \) is \( \overline{B} = \{ \tau \in K \mid \exists \sigma \in B \, \tau \leq \sigma \} \).

• The star of \( B \subseteq K \) is \( \text{St}_B = \{ \tau \in K \mid \exists \sigma \in B \, \tau \geq \sigma \} \).

• The link of \( B \subseteq K \) is \( \Lambda_B = \text{St}_B - \overline{B} \).

• The closure of \( B \subseteq K \) is \( \overline{B} = \{ \tau \in K \mid \exists \sigma \in B \, \tau \geq \sigma \} \).

• Two edge contractions in a 2-manifold are independent if they do not affect the same triangle. That is, \( \text{St}_{ab} \cap \text{St}_{cd} = \emptyset \).

• A triangulation of a 2-manifold is irreducible if no edge is contractible.

• Two edge contractions in a 2-manifold are independent if they do not affect the same triangle.

A triangulation of a 2-manifold is irreducible if no edge is contractible.

An edge is contractible if its contraction does not change the surface topology.

• The link of \( B \subseteq K \) is \( \Lambda_B = \{ \tau \in K \mid \exists \sigma \in B \, \tau \neq \sigma \} \).

• The star of \( B \subseteq K \) is \( \text{St}_B = \{ \tau \in K \mid \exists \sigma \in B \, \tau \geq \sigma \} \).

Basic definitions
Results overview

- Topology Preserving Edge Contractions, by Dey, Edelsbrunner, Guha, and Nekhayev, 1999.
  - Results in characterization of topological preserving edge contractions for simplicial complexes up to dimension $3$.
  - Improved bound on the number of vertices in an irreducible triangulation of an orientable $2$-manifold $[2]$.
  - Computing a topological preserving hierarchy of $O((b + u)\log u + b)O$ depth for an orientable $2$-manifold $[2]$. Size and edge contractions in an orientable $2$-manifold, each of which effect a small number of triangles, of which a greedy algorithm to find $\Theta(n)$ independent topological preserving edge contractions for simplicial complexes up to dimension $3$ [1].
- Results in characterization of topological preserving edge contractions by Dey, Edelsbrunner, Cuchta, and Nekhayev, 1999.
- Topology Preserving Edge Contractions, by Dey, Edelsbrunner, Cuchta, and Nekhayev, 1999.

A vertex map for two complexes \( K \) and \( L \) is a map \( f : \text{Vert } K \rightarrow \text{Vert } L \).

The simplicial map \( \phi \) for a vertex map is defined by

\[
(n) f \cdot (x)^n q \begin{array}{c}
\sum \\
\end{array} \begin{array}{c}
\text{Vert } K
\end{array} = (x) \phi
\]

\[
(x)^n q \begin{array}{c}
\sum \\
\end{array} \begin{array}{c}
\text{Vert } K
\end{array} = 1
\]

\[
n \cdot (x)^n q \begin{array}{c}
\sum \\
\end{array} \begin{array}{c}
\text{Vert } K
\end{array} = x
\]

The barycentric coordinates of a point \( x \in \text{Vert } K \), \( \phi \in \text{Vert } \) \( L \), and \( L \subseteq n \) if \( n \neq (x)^n q \) so only \( n \in \text{Vert } K \), \( x \in \text{Vert } \) \( L \), are the unique simplicial map.

A vertex map for two complexes \( K \) and \( L \) is a map.

\[\leftarrow \begin{array}{c}
\text{Vert } K
\end{array} \]
Unfoldings

- $f: \mathcal{K} \rightarrow \mathcal{L}$ is a simplicial homeomorphism if and only if it is bijective and $f^{-1}$ is also a vertex map.

- An edge contraction of $\mathcal{K}$ is a simplicial map $\phi_{ab}: \mathcal{K} \rightarrow \mathcal{L}$ defined by a surjective vertex map $f(u) = \begin{cases} u & \text{if } u \in \text{Vert } \mathcal{K} - \{a,b\} \\ c & \text{if } u \in \{a,b\} \end{cases}$.

- Note that outside $\mathcal{K}_{ab}$, $\phi_{ab}$ is the identity, but inside it is not even injective.

- An unfolding of $\phi_{ab}$ is a simplicial homeomorphism $\psi: \mathcal{K} \rightarrow \mathcal{L}$.

- $\psi$ is a local unfolding if it differs from $\phi_{ab}$ only inside $\mathcal{K}_{ab}$.

- $\psi$ is a relaxed unfolding if it differs from $\phi_{ab}$ only inside $\mathcal{K}_{ab}$.

- An edge contraction of $\mathcal{K}$ can then be defined as a surjective simplicial map $\mathcal{T} \leftarrow |Y|: q \circ \phi$.

- Unfoldings $\psi$ is a simplicial homeomorphism if and only if $\mathcal{T} \leftarrow |Y|: \psi$.
The order of $\sigma$ is the smallest integer $i$ for which there is a $(k-i)$-simplex $\eta$ such that $\sigma$ and $\eta$ are combinatorially equivalent.

Since the interior of $\mu$ is homeomorphic to $\mathbb{R}^{k-i}$, the star of $\sigma$ is homomorphic to $\mathbb{R}^{k-i}$ for some topological space $X$ of dimension $k-i$.

The $j$-th boundary of a simplicial complex $K$ is the set of simplices

$$\{ \ell \leq \sigma \mid X \ni \ell \} = \text{Bd}_j K$$

with order no less than $j$.

**Order and Boundary**
For a 1-complex $K$, the following are equivalent:

1. $\partial_0 a \cap \partial_0 b = \emptyset$.
2. $\phi_{ab}$ has a local unfolding.
3. $\phi_{ab}$ has an unfolding.

For a simplex $\sigma \in \partial_i K$, we denote the link within $\partial_i K$ as $L_{\partial_i K} \sigma$.

For a simplex $\sigma \in \partial_i K$, we denote the link within $\partial_{i+1} K$ by $L_{\partial_{i+1} K} \sigma$.

$\partial_{i+1} K \cap \partial_i K = \partial_{i+1} K$, and comes from $\sigma$ to all simplices in the $(i+1)$-th boundary.

For each $i$, we extend the $i$-th boundary by adding a dummy vertex.

1-complexes
For a 2-complex $K$ then the following statements are equivalent:

1. (i) $L_k \omega a \cap L_k \omega b = L_k \omega ab$, and $L_k \omega 1 a \cap L_k \omega 1 b = \emptyset$.
2. (ii) $\varphi_{ab}$ has a local unfolding.

They demonstrate a 2-complex which has neither a local nor a relaxed unfolding.

For a 2-manifold the following statements are equivalent:

1. (i) $L_k a \cap L_k b = L_k ab$.
2. (ii) $\varphi_{ab}$ has a local unfolding.
3. (iii) $\varphi_{ab}$ has an unfolding.

They demonstrate a 2-complex which has neither a local nor a relaxed unfolding.
Steinitz' Theorem (1922): For every 3-connected planar graph \( G \), there is a convex 3-polytope with an isomorphic 1-skeleton.

• A convex 3-polytope with an isomorphic 1-skeleton.

• A graph \( G \) is 3-connected if the deletion of any two vertices together leaves the graph connected.

• A graph \( G \) is planar if it is isomorphic to a 1-complex in \( \mathbb{R}^2 \).

• The 1-skeleton is the subcomplex of all vertices and edges.

• That do not all lie in a common plane.

• A convex 3-polytope is the convex hull of finitely many points in \( \mathbb{R}^3 \).
For a 3-complex $K$ the following statements are equivalent:

1. (i) $L_k \omega^0 a \cap L_k \omega^0 b = L_k \omega^0 ab$,
2. (ii) $\phi^{ab}$ has a relaxed unfolding.
3. (iii) $\phi^{ab}$ has a local unfolding.

For a 3-manifold the following statements are equivalent:

1. (i) $L_K a \cap L_K b = L_K ab$.
2. (ii) $\phi^{ab}$ has a local unfolding.
3. (iii) $\phi^{ab}$ has an unfolding.

\[ \emptyset = q^{-1}_K a \cup q^{-1}_K b. \]
\[ L_K a \cap L_K b = q^{-1}_K a \cap q^{-1}_K b. \]
Topology preserving edge contractions by Dey, Edelsbrunner, Guha, and Nekhayev, 1999.

Results in characterization of topological preserving edge contractions for simplicial complexes up to dimension 3 [1].


Hierarchy of Surface Models and Irreducible Triangulation by Ghareh, and Nekhayev, 1999.
References
