TOPOLOGY OF POINT CLOUD DATA

CS 468 – Lecture 8
11/12/2
PROJECTS

• Writeups:
  – Introduction to Knot Theory (Giovanni De Santi)
  – Mesh Processing with Morse Theory (Niloy Mitra)

• Presentations:
  – November 27th:
    * Surface Flattening (Jie Gao)
    * Simplicial Sets (Patrick Perry)
    * Complexity of Knot Problems (Krishnaram Kenthapadi)
  – December 4th
    * Discrete Morse Theory(?) (Yichi Gu)
    * Irreducible Triangulations (Jon McAlister)
    * Homotopy in the Plane (Rachel Kolodny)
OVERVIEW

• Points
• Complexes
  – Čech
  – Rips
  – Alpha
• Filtrations
• Persistence
POINTS

- $m$ samples $M = \{m_1, m_2, \ldots, m_m\}$ from a manifold $\mathbb{M}$
- Samples are embedded, but intrinsic topology is lost
- Error: acquisition device noise and approximation
POINT CLOUD DATA

(a) Surface  (b) Molecule  (c) Universe
• **$\epsilon$-ball**: $B_\epsilon(x) = \{y \mid d(x, y) < \epsilon\}$.

• Open sets and topology

• Manifold is $\tilde{M} = \bigcup_{m_i \in M} B_\epsilon(m_i)$
ČECH COMPLEX

\[ C_\varepsilon(M) = \left\{ \text{conv } T \mid T \subseteq M, \bigcap_{m_i \in T} B_\varepsilon(m_i) \neq \emptyset \right\}. \]

\[ \sum_{k=0}^{m} \binom{m}{k} = 2^{m+1} - 1 \]

\[ C_\varepsilon(M) \simeq \tilde{M} \]
RIPS COMPLEX

- $R_\epsilon(M) = \{\text{conv } T \mid T \subseteq M, d(m_i, m_j) < \epsilon, m_i, m_j \in T\}.$
- Still $O\left(\binom{m}{k}\right)$ for the $k$th skeleton
- Need $(k + 1)$st skeleton for computing $H_k$
• \( V(m_i) = \{ x \in \mathbb{R}^3 \mid d(x, m_i) \leq d(x, m_j) \forall m_j \in M \} \)

• \( \hat{V}(m_i) = B_\epsilon(m_i) \cap V(m_i) \)

• \( A_\epsilon = \left\{ \text{conv } T \mid T \subseteq M, \bigcap_{m_i \in T} \hat{V}(m_i) \neq \emptyset \right\} \)

• \( A_\epsilon(M) \simeq \tilde{M}, A_\epsilon \subseteq D, \text{ the Delaunay complex} \)

• \( O(n \log n + n^{\lceil d/2 \rceil}) \)
• Extendible to points with weights
• van der Waals model of molecules
Filtrations

- Complexes $C_\epsilon, R_\epsilon, A_\epsilon$, compute homology!
- Which $\epsilon$? Vary and get a filtration!
- A filtration of a complex $K$ is $\emptyset = K^0 \subseteq K^1 \subseteq \ldots \subseteq K^m = K$. 
Bunny

- 34,834 points, 1,026,111 complexes
• 312 atoms, 8,591 complexes
**Approach**

- Input: point cloud
- Procedure:
  - Put $\epsilon$-balls around points
  - Compute complex $K_\epsilon$
  - Compute homology of complex
- Varying $\epsilon$ gives us a filtration
- Incremental algorithm gives homology of filtration (demo)
HOMOLOGY OF A FILTRATION

• $K^l$ is a filtration

• $Z_k^l = Z_k(K^l)$ and $B_k^l = B_k(K^l)$ are the $k$th cycle and boundary group of $K^l$, respectively.

• The $k$th homology group of $K^l$ is $H_k^l = Z_k^l/B_k^l$.

• The $k$th Betti number $\beta_k^l$ of $K^l$ is the rank of $H_k^l$. 
PROBLEM

- **Features**
- **Noise**: spawned by noise, representation, etc.
**PERSISTENCE**

- $K^l$ be a filtration.

- The $p$-persistent $k$th homology group of $K^l$ is
  \[
  H^{l,p}_k = \mathbb{Z}_k^l / (B^{l+p}_k \cap \mathbb{Z}_k^l),
  \]

- The $p$-persistent $k$th Betti number $\beta^{l,p}_k$ of $K^l$ is the rank of $H^{l,p}_k$.

- Well-defined

- $\eta^{l,p}_k : H^l_k \rightarrow H^{l+p}_k$,

- $\text{im} \ \eta^{l,p}_k \cong H^{l,p}_k$.

- This lecture: $\mathbb{Z}_2$ homology
Lifetimes

• Let $z$ be a non-bounding $k$-cycle, created when $\sigma$ enters complex at time $i$

• That is, $\beta_{k++}$ at time $i$

• $z$ creates a class of homologous cycles $[z]$

• $[z]$ is merged with the boundary class at time $j$ when $\tau$ enters ($\beta_{k--}$)

• $\tau$ destroys $z$ and the cycle class $[z]$.

• The persistence of $z$, and its homology class $[z]$, is $j - i - 1$.

• $\sigma$ is the creator (positive) and $\tau$ is the destroyer (negative) of $[z]$.

• If a cycle class does not have a destroyer, its persistence is $\infty$. 
LIFETIME REGIONS

- \( H_{k}^{l,p} = Z_{k}^{l} / (B_{k}^{l+p} \cap Z_{k}^{l}) \)
- Basis element \( z + B_{k} \) lives during \([i, j)\)
- \( z \notin B_{k}^{l} \) for \( l \leq j \)
- Therefore, \( z \notin B_{k}^{l+p} \) for \( l + p < j \).
- \( p \geq 0 \)
- \( l \geq i \)
• $p \geq 0$
• $l \geq i$
• $l < j$
TRIANGLES
GRAPH OF $\log(\beta_{1, p}^l + 1)$
CMY Color Space

- Green
- Yellow (minus blue)
- Cyan (minus red)
- Black
- Red
- Blue
- Magenta (minus green)
Topology Maps
Algorithm

- Compute $\partial_k \sigma_i$
- Eliminate negative simplices in chain
- Look for youngest cycle creator and store
We can compute:

- Cycles (components, cycles, voids)
- Bounding manifolds

(Demo)

Points can be anything

- samples from high dimensional manifolds: configuration spaces for robots (PRM), time-variant data, etc.
- samples of tangent complex for data-set $\mathbb{M} \times S^2$

Need fast $d$-dim complex builder