CS368: Geometric Algorithms Handout #14
Design and Analysis
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Voronoi Diagrams and Delaunay Triangulations

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This lecture deals with two methods of solving proximity problems: Voronoi diagrams and Delaunay Triangulation. These fundamental diagrams are duals of one another. We will begin by presenting some preliminary facts about these diagrams.

1 Voronoi Diagrams

We are given n *sites* (points) in the plane; for any given point, we wish to know which site (or sites) is closest. This problem has been referred to as the "post-office problem" in Knuth, Volume 1. It generates a subdivision of the plane, with each region corresponding to the locus of all points closest to a particular site. This partitioning of the plane is called the Voronoi diagram.

It is worth considering a few simple cases (see figure 1). If there is only one site, then the entire plane is one region. If there are two distinct sites A and B, then there are three distinct regions: the points equidistant from the two sites (the perpendicular bisector of the segment between the sites), the locus of points closest to A, and the locus of points closest to B. If there are three sites (not collinear), we have three unbounded regions divided by three rays from a central vertex (point); this central vertex is equidistant from all three sites (i.e., it is the center of the circle through the sites), and the rays lie on the bisectors of the segments joining the sites. If there are four sites, there are two topologically distinct (and nondegenerate[1]) cases; if the four sites are in convex position, then all four regions are unbounded. If one of the four sites is surrounded by the other three, then there is one bounded face (a triangle) corresponding to the surrounded site.

What distinguishes these two cases? Note that sites on the convex hull of the sites give rise to infinite regions, while those sites within the convex hull correspond to finite regions.

Claim 1. *A site x has an unbounded region if and only if x lies on the convex hull of the sites.*

*based on notes by Michelangelo Grigni at MIT

[1] Loosely speaking, the arrangement of sites is nondegenerate if no sufficiently small perturbation of the sites changes the topology of the diagram. For Voronoi and Delaunay diagrams non-degeneracy means that no two sites are identical, no three sites are collinear, and no four sites are co-circular.

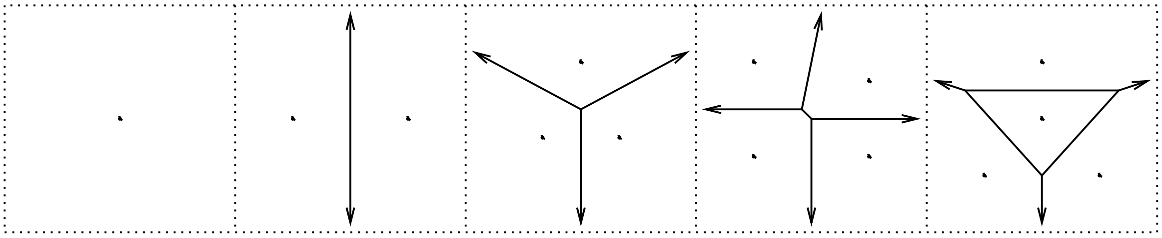

Figure 1: Some simple Voronoi diagrams for $n \leq 4$ sites.

Proof: If x is a site on the convex hull, then we may take a supporting line l through x such that all the sites lie on the same side of l. Then every point on a ray normal to l going away from the hull has x as its nearest point; hence, the Voronoi region of x is unbounded since it contains this ray. (Note this argument even applies in the degenerate case when there are three collinear sites on the hull).

Conversely, if site x has an unbounded region, then since the region is convex, it must contain an infinite ray emanating from x. Taking the line l through x normal to this ray, we see that every other site y must lie on the other side of l (otherwise by going out sufficiently far on the ray, we would find a point closer to y than to x). $\square$

Claim 2. *All the Voronoi regions are convex.*

Proof: Fix a site x. For every other site y, the Voronoi region of x is contained in the half-plane of points that are at least as close to x as they are to y. Thus, the Voronoi region of x is exactly the intersection of all such half-planes for different sites y. Since each half-plane is convex and the intersection of convex regions is also convex, the Voronoi region of x is convex. Note this argument also shows that the boundary of the Voronoi region of x is polygonal (i.e., it is bounded by line segments or rays). $\square$

Note that each vertex or edge of the Voronoi diagram depends on only a bounded number of the sites: a vertex is defined by its three nearest sites, and an edge is defined by four sites – the two sites on whose bisector it lies, and two other sites delimiting its extent. This "finite dependency" property will arise in many other settings in geometric algorithms. Thus, if we compute the coordinates of one of the vertices of the diagram, those coordinates are low degree rational functions in the coordinates of the three sites that define it. But still it would be useful to compute a diagram with no "new" real numbers, and this motivates the Delaunay diagram.

2 Delaunay Triangulations

[2]This figure swiped from page 105 of [GS] (i.e. Handout 12).

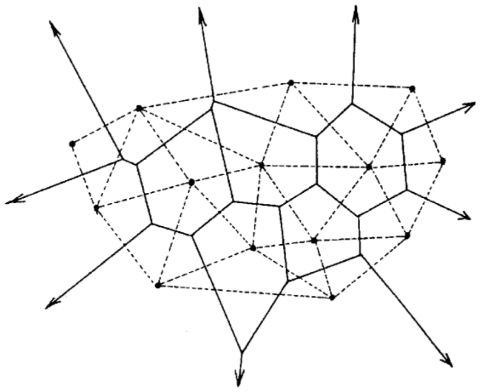

Figure 2: The Voronoi diagram (solid) and the Delaunay diagram (dashed).[2]

The Delaunay triangulation is the *graph theoretic dual, topological dual,* or the *combinatorial theoretic dual* of the Voronoi diagram (see figure 2): the vertices of the Delaunay triangulation are the sites (corresponding to the regions of the Voronoi diagram), and we connect two sites with an edge in the Delaunay triangulation iff their Voronoi regions share an edge (thus, the two diagrams have the same number of edges, although the Delaunay triangulation has no unbounded edges). Thus, a vertex in the Voronoi diagram corresponds to a Delaunay triangle, a Voronoi edge to a Delaunay edge, and a Voronoi face to a Delaunay vertex (site). Note, however, that dual edges of the two graphs do not necessarily intersect. Furthermore, we may recover the Voronoi diagram by reversing the process. Note there are no 'new' real numbers needed to describe the Delaunay triangulation.

We now prove various simple properties of the Delaunay triangulation. In particular, we present the notion that site-free circles through two or three sites are "witnesses to Delaunay-hood" for edges or triangles, respectively.

Claim 3. *For every edge ab in the Delaunay triangulation, there is a circle, called a* witness, *through sites a and b that contains no other site. The converse is also true.*

Proof: Since a and b are connected, their Voronoi regions must share an edge. Take a point p in the interior of this edge, then a and b are the same distance r from p, and all other sites are further away. Hence, the circle of radius r about p suffices.

Conversely, the existence of a circle through a and b with all other sites strictly outside implies that the center of the circle is on a boundary edge between the Voronoi regions of a and b. □

Claim 4. *The Delaunay triangulation is planar.*

Proof: It suffices to prove that no two edges may cross. Suppose edges ab and xy appear in the Delaunay triangulation, and these edges cross. Then there is a circle through a and b not containing x or y, and likewise there is a circle through x and

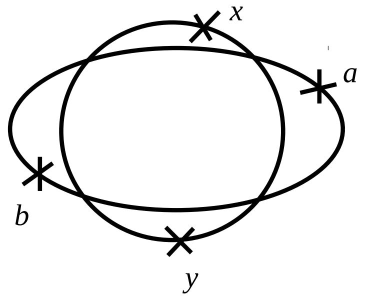

Figure 3: Two circles cannot cross at 4 points.

y not containing a or b. But then these two distinct circles must intersect at four points, which is impossible. See figure 3. □

Various other properties may also be argued. If abc is a triangle in the Delaunay triangulation, then no other site lies inside its circumcircle, and conversely.

We now turn to counting problems. We will assume non-degeneracy (i.e., no four points are co-circular, no three points are collinear). We will make precise claims about the number of edges and triangles in the triangulation.

Let n be the number of sites, t the number of triangles, and e the number of edges. From Euler's Theorem for connected planar graphs, we have

$$n - e + t + 1 = 2$$

or

$$n - e + t = 1$$

where $t+1$ is the number of faces (each triangle is a face and so is the single unbounded face). We need one more parameter to describe the arrangement. Since each edge belongs to two faces, we have $3t + k = 2e$, where k is the number of sites on the convex hull. Solving, we have

$$\begin{aligned} e &= 3(n-1) - k, \\ t &= 2(n-1) - k. \end{aligned}$$

By duality, the Voronoi diagram will have e edges (bounded or unbounded) and t vertices. Hence, these graphs have only linear complexity in the size of the input, which will be very important to the computational complexity of various planar proximity problems. For example, this linear complexity implies that the corresponding data structures can be stored in $O(n)$ space and that the arrangement can be computed in $O(n \log n)$ time. However, this only holds for two dimensions. For the d-dimensional case, we have $O(n^{\lfloor \frac{d+1}{2} \rfloor})$ complexity.

At this point, it was stated that the Delaunay triangulation was more intuitive than the Voronoi diagrams, so we will tend to use the Delaunay triangulation in this

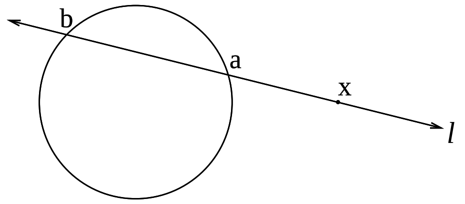

Figure 4: The power of point x with respect to the circle is $ax \cdot bx$, which is independent of the choice of line l.

course. Another comment: Since the triangulation starts with numerical data and ends up with combinatorial output (e.g. a graph), complications may arise due to round-off errors, etc.

Delaunay triangulations are used in many applications (e.g., finite element analysis, interpolation). This triangulation is especially useful because it is easy to compute and it provides a "nice" triangulation (i.e., the smallest angle in the triangulation is the largest possible).

Suppose we have n sites, and someone claims a given triangulation is a Delaunay triangulation. How much work do we have to do to verify the claim? If for each site and each triangle we verify that the site is not in the circumcircle of the triangle, then this is clearly sufficient. Unfortunately, this is an $O(n^2)$ time algorithm.

Delaunay [D] proved a remarkable theorem, that it suffices to only check adjacent triangles, so that the verification may be performed in linear time.

Theorem 5. *Given a triangulation of n sites such that for every pair of adjacent triangles abc and bcd, a is not in the circumcircle of bcd, then that triangulation is the Delaunay triangulation.*

Proof: First, we define the "power" of a point with respect to a circle. Given a point x and a circle C, draw a line l through x that intersects C at two points a and b. Then the power of the point is $xa \cdot xb$ (here xa denotes the distance from x to a). It turns out that this quantity is independent of the line chosen. Furthermore, given a suitable choice of signs for distances along the line, the sign of the power determines whether x is inside, outside, or on the circle (the usual convention makes the power positive iff x is outside the circle).

Take a triangle abc of the given triangulation and a site x that lies on the opposite side of (say) line bc. If the neighboring triangle to abc along bc is xbc, then we are done by assumption.

Otherwise, consider the line connecting site x to a point d in the interior of triangle abc. As we move along xd from x to d, we pass through a sequence of triangles, starting with a triangle having x as a vertex (call it vwx) and ending with triangle abc.

Now consider the sequence of circumcircles of these triangles. We start with the circumcircle of vwx where the power of x is 0 (since x is on the circle). We claim that the power of x can only increase as we move to circumcircles of triangles further

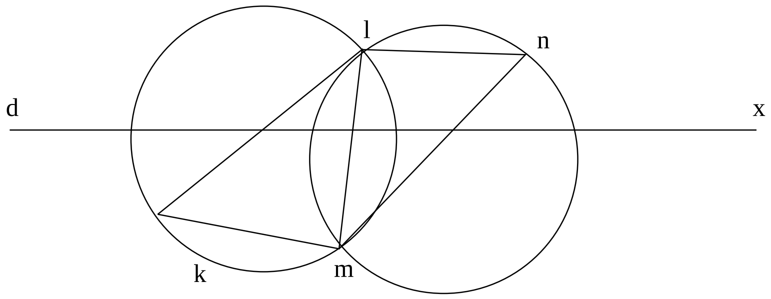

Figure 5: The power of x with respect to lmn is less than the power of x with respect to klm.

away from x. This claim implies that the power of x with respect to the circumcircle of abc must be positive, so that x is not in the circumcircle of abc, which is what we wanted to prove.

We now argue our claim informally. Given adjacent triangles lmn and klm on the line from x to d (see figure 5), consider their circumcircles. These circumcircles are both members of the family of circles through l and m, and we may continuously "push" the circumcircle of lmn away from site n until it reaches site k. During this push operation, both intersection points of the circle with xd move further away from x, hence the power of x increases. □

References

[D] B. Delaunay, "Sur la Sphère Vide," *Izvestia Akademii Nauk SSSR, VII Seria, Otdelenie Matematicheskii i Estestvennyka Nauk*, 7(6): 793–800, October 1934. English translation by M. Dillencourt, manuscript.

[GS] Leonidas Guibas and Jorge Stolfi. "Primitives for the manipulation of general subdivisions and the computation of Voronoi diagrams." *ACM Transactions on Graphics*, 4(2):74–123, April 1985.