CS448: Topics in Computer Graphics Homework #1
Mathematical Models for Computer Graphics
Stanford University October 9

Homework #1: Wavelets
Due Date: October 23

1. Let

$$X(\omega) = \begin{cases} 1, & \text{if } |\omega| < \pi \\ 0 & \text{otherwise} \end{cases}$$

What function $x(t)$ has Fourier transform equal to $X(\omega)$? This function is a solution of a dilation equation

$$\phi(t) = \sum_k h(k)\phi(2t-k)$$

find the coefficients $h(k)$. Using alternating flip, i.e. $h_1(n) = (-1)^n h_0(-n+1)$, find the coefficients and explicit form of the corresponding wavelet. *Hint:* write the alternating flip condition in the Fourier domain.

2. Show that a filter $h_0(n)$ of order N, i.e. having $N+1$ non-zero coefficients $h(0)$, $h(1)$,... $h(N)$ generates a scaling function supported on $[0 \dots N]$; assume that the cascade algorithm for this filter converges pointwise.

3. In all orthogonal filter banks that we have considered the highpass filter coefficients $h_1(n)$ were obtained from lowpass filter coefficients by alternating flip. Recall that N for orthogonal filter banks has to be odd. In this problem we show that all possible highpass filters for a given lowpass filter are just even translates of the filter obtained by alternating flip. Show that for an orthogonal filter bank

$$h_1(n) = \pm(-1)^n h_0(2k-1-n)$$

for some integer k.

You may assume that both filters are finite. *Hint:* Recall that the action of the filter bank on a signal can be thought of as a linear transform. We can change the order of elements of the input vectors putting all even-numbered elements first and all odd-numbered elements after them. Rewrite the filter bank matrix $[LB]^T$ separating even coefficients from odd:

$$
\left[\begin{array}{cccccccc}
. & . & & & & & & \\
. & c(3) & c(2) & c(1) & c(0) & . & . & \\
. & . & . & c(3) & c(2) & c(1) & c(0) & . \\
 & & & & . & . & . & . \\
\hline
. & . & & & & & & \\
. & d(3) & d(2) & d(1) & d(0) & . & . & \\
. & . & . & d(3) & d(2) & d(1) & d(0) & . \\
 & & & & . & . & . & .
\end{array}\right]
$$

becomes

$$
\left[\begin{array}{ccccc|ccccc}
. & . & & & & . & . & & & \\
. & c(2) & c(0) & . & . & . & c(1) & c(3) & & \\
. & . & c(2) & c(0) & . & . & . & c(1) & c(3) & . \\
 & & & . & . & & & . & . & . \\
\hline
. & . & & & & . & . & & & \\
. & d(2) & d(0) & . & . & . & d(1) & d(3) & & \\
. & . & d(2) & d(0) & . & . & . & d(1) & d(3) & . \\
 & & & . & . & & & . & . & .
\end{array}\right]
$$

In matrix form, this means that

$$
\begin{bmatrix} L \\ B \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} C_e & C_o \\ D_e & D_o \end{bmatrix}
$$

where C_e, C_o, D_e, D_o are matrices with rows being the even/odd coefficients of the filters. Note that the shift between sequential rows is 1, not 2 as it was in the original matrix. Show that for an orthogonal filter bank $C_o C_o^T = D_e D_e^T$ and $C_e C_e^T = D_o D_o^T$. Rewrite these expressions in terms of the filter coefficients. Recall that a shift by k becomes multiplication by $e^{i\omega k}$ after the Fourier transform is applied. Use this fact to derive the equations relating $d(n)$ and $c(n)$ in the Fourier domain. For given $c(n)$ find all possible $d(n)$ satisfying these equations. By substitution, show that the only solutions that result in orthogonal filters are the ones obtained by alternating flip with even shift and possibly a sign change.

4. It is possible to construct new scaling functions out of the ones that we already have. One particular way to construct new scaling function with better properties (higher smoothness in particular) is to use convolution. Show that if $\phi_1(t)$ and $\phi_2(t)$ are solutions of scaling equations with coeficients $h(n)$ and $h'(n)$, then $\int \phi_1(t-s)\phi_2(s)ds$ is also a solution of a scaling equation. Find coefficients of this equation.

5. The key to construction wavelets is to choose the filter coefficients in the dilation equation. Then the scaling function can be obtained by using the cascade algorithm. Unfortunately, the algorithm does not always converge: it is necessary for convergence that coefficients sum up to 1, but this is not sufficient. There is no

universal sufficient condition for pointwise convergence and filters have to be analyzed on case by case basis. It is particularly important to establish convergence for biorthogonal constructions, when the coefficients for the analysis lowpass filters are derived from coefficients for lowpass synthesis filters and the resulting cascade algorithm for analysis does not converge.

a. Find an example of a filter for which the cascade algorithm diverges pointwise, but the coefficients of the filter add up to 1 and are all positive.

b*. Find an example of a filter for which the cascade algorithm diverges in L^2 norm.

6. * **Heisenberg inequality.** We have discussed in the class that wavelet bases combine spatial and frequency localization; sometimes it would be desirable to have arbitrarily good localization both in frequency and in time, that is, both a basis function and its Fourier transform are significantly different from zero only on very short intervals. It turns out that this is impossible; impossibility of a quantum particle having precise momentum and position at the same time is a consequence of this mathematical fact.

Define RMS (root-mean squared) duration of a continuous signal $x(t)$ as

$$D_t = \left(\frac{1}{E_t}\int_{-\infty}^{\infty} t^2 x^2(t)dt\right)^{\frac{1}{2}}$$

where

$$E_t = \int_{-\infty}^{\infty} x^2(t)dt$$

Similarly, D_ω is defined as the RMS duration of the Fourier transform $X(\omega)$. For simplicity, we will assume that the Fourier transform of the signal is real. Show that

$D_t D_\omega \geq \frac{1}{2}$

Use the relation between the Fourier transform of a function and its derivative and Cauchy-Schwartz inequality:

$$\left(\int u(t)v(t)dt\right)^2 \leq \int u^2(t)dt \int v^2(t)dt$$

where the functions are integrated over the whole real line.

You will also need Parseval's identity:

$$\int_{=-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^2(\omega)d\omega$$

Assume that $tx(t) \to 0$ as $t \to \infty$ and all necessary integrals exist.

7. **Oversampled filter banks.**

Recall that subsampling of a signal introduces aliasing if the signal is not band-limited. For the low-pass component of a signal that passed through a filter bank aliasing is relatively small (the signal is close enough to band-pass.) Aliasing is more significant in the highpass component. Perfect reconstruction us achieved when aliases in two channels cancel each other.

In an oversampled filter bank the high-frequency component of the signal is not subsampled. As a result, aliasing is avoided, but a finite signal of length n is transformed into a signal of length $1.5n$. In some cases oversampled filter banks have some advantages over critically subsampled filter banks used for wavelet transforms. This happens when it is necessary to modify the high-pass and low-pass components of the signals independently, for example, for multiresolution editing of surfaces or images. Other examples of applications of oversampled filter banks include Burt-Adelson pyramid and similar mipmap pyramids.

In this problem we show when the two approaches can be combined: we can easily switch between oversampled and critically subsampled representations.

An oversampled filter bank works as follows. First, a low-pass filter H_0 and down-sampling is applied to a signal x to produce a low-pass component x_0; then the signal is upsampled and the lowpass filter is applied again to produce a smoothed version of the original signal x'. Finally, the difference $d = x - x'$ is used as the second component of the signal; i.e. the transformed signal is (x_0, d). Clearly, the signal x can be reconstructed in unchanged form from (x_0, d) (perfect reconstruction).

a. Find the expressions for x_0 and d in terms of the original signal x and the filter matrix H_0; rewrite x_0 and d in terms of filter coefficients $h_0(n)$. Assume that x is finite and for the purposes of filtering x is extended by zeroes whenever the filter requires values of the signal before the first and after the last.

b. Express the reconstruction of the signal x from x_0 and d in matrix form.

c. The transform described above can be applied recursively to the lowpass component of the signal; what will be the total length of the transformed signal of we start with a signal x of length 2^p and apply the transform recursively until we obtain the signal of length 1?

d. Show that if

$$\sum h_0(k)h_0(k-2n) = 0 \text{ for all } n$$

then d can be filtered using $h_1(n)$, subsampled, and then prefect reconstruction still is possible with a suitable filter bank (use alternating flip.)