**CS448: Topics in Computer Graphics** Homework #3
**Mathematical Models for Computer Graphics**
Stanford University December 3

Homework #3: Subdivision
Due Date: Monday, December 15, Noon
**No extensions are possible, grades are due Dec. 16!**

Problems marked with * are optional problems. Extra credit will be given for these problems.

1. **Polynomial reproduction.** Recall that the four-point interpolating subdivision scheme for curves has coefficients $-1/16$, $9/16$, $9/16$, $-1/16$ for odd points. The limit curves produced by this scheme need not be piecewise polynomials. However, this scheme has a remarkable property relating it to polynomials.

   Prove that if all control points $c_i$ ($i$ is an integer), have coordinates $c_i^x = p^x(i)$, $c_i^y = p^y(i)$, where $p^x$ and $p^y$ are polynomials of degree no greater than 3, then the limit curves generated by the four-point scheme are polynomial curves $(p^x(t), p^y(t))$.

   *Note:* This is a specific example of a more general property of subdivision: if the curves or surfaces generated by subdivision are $C^k$-continuous, the scheme should reproduce polynomials of degree at least up to $k+1$.

2. **Convergence of subdivision.** As we have discussed in class, once we have a convergent subdivision scheme for curves, we can easily obtain a scheme generating curves with arbitrary number of derivatives. However, it is difficult to construct convergent schemes. Typically, coefficient for a specific scheme are guessed in some way and then convergence is proved. In this problem we prove that the four-point scheme converges; the method that we use works for a variety of schemes.

   a. Recall that coordinates of control points on levels $j$ and $j+1$ are related by

   $$x_i^{j+1} = \sum_k s_{2k-i} x_k^j$$

   ($x_i^j$ is one of the $x, y$ coordinates of a control point, a single number, not a 2D point.) Find a formula of the same form relating the *differences* $x_{i+1}^{j+1} - x_i^{j+1}$ to $x_{i+1}^j - x_i^j$ for the four-point scheme.

   b. Deduce from the recursive formula for differences that the differences converge to zero as $j \to \infty$. This is the crucial element of the convergence proof.

   c*. Assuming that $\max_i |x_{i+1}^0 - x_i^0|$ is bounded by a constant $C$, show that the sequence of piecewise-linear approximations generated by subdivision converges pointwise, that is, for any fixed $t$ the linear approximation $|L^j(t) - f(t)|$ converges to zero. $L^j(t)$ is the piecewise-linear approximation to the limit subdivision curve $f(t)$. *Hint.* Estimate the difference $|L^{j+1}(t) - L^j(t)|$ and observe that $|L^j(t) - f(t)| =$

$\sum_{i=j}^{\infty}(L^i(t) - L^{i+1}(t))$. *Note.* The estimates derived in the last part of the proof can be used to estimate the precision of linear approximation $L^j(t)$.

3. **Piecewise-smooth Catmull-Clark.** We have seen how it is possible to modify the subdivision rules for Loop's scheme to add sharp creases to a surface. Describe a similar modification of rules for the Catmull-Clark scheme.

4. **Adaptive subdivision.** One of the attractive features of subdivision is that it is possible to refine curves or surfaces adaptively. For example, we can subdivide more in places where the curve varies a lot, and subdivide less in places where it is almost flat. However, we might need to compute some extra control points which are not necessary to approximate the curve well according to our error criterion, but are required to compute points on finer levels. In this problem we find in a specific case how many extra control points have to be computed.

   Suppose we are approximating a degree 5 B-spline curve with 8 control points $c_i$ $i = 0\dots 7$. Suppose that we want to subdivide the interval between control points $c_3$ and $c_4$ four times, and do not want to subdivide other intervals at all, unless it is required to subdivide the interval of interest. What control points outside the interval between $c_3$ and $c_4$ we have to compute? How many additional points have to be computed?

5. **Subdivision matrix for Loop's scheme.** The subdivision matrix plays an important role in analysis of subdivision. The subdivision matrix can be used to compute tangent vectors of subdivision surfaces, which allow us to compute the exact normal to the surface. Recall that the tangent vectors are given by the coefficients $a_1$ and $a_2$ in the decomposition of the vector of control vertices with respect to the eigenbasis of the subdivision matrix.

   It turns out that to compute $\mathbf{a}_1$ and $\mathbf{a}_2$ we do not need the eigenvectors of the whole subdivision matrix, it is sufficient to consider a smaller submatrix, acting on the subset of the control vertices that were used to define subdivision matrix. In this problem we derive a general form for this submatrix for Loop's scheme using the Discrete Fourier transform (DFT).

   For a finite vector $y_0, y_1 \dots y_{n-1}$ the DFT of this vector is another vector of the same length $\hat{y}_0, \dots \hat{y}_{n-1}$ defined as

$$\hat{y}_j = \sum_{k=0}^{n-1} w^{jk} y_k$$

   where $w = e^{-2i\pi/n}$. The properties of DFT are similar to the properties of the regular Fourier transform, but there is a slight difference. Recall that the Fourier transform of a shifted signal differs from the Fourier transform of the original signal by a factor $e^{2i\omega k}$. Let shift$(y,k)$ be the *circular shift* of the sequence $a$, defined by $shift(y,k)_i = y_{(i-k) \bmod n}$. Then for the $j$-th component of DFT we have

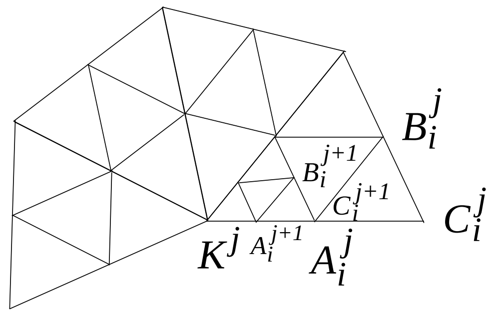
$$[\mathrm{DFT}((\mathrm{shift}(y,k))]_j = \omega^{jk}[\mathrm{DFT}(y)]_j$$

Also note that DFT is a linear transformation and can be written as multiplication by a matrix $W$:

$$\hat{y} = Wy$$

where $(W)_{jk} = e^{-2ijk\pi/n}$.

a. The subdivision matrix for Loop's scheme acts on vectors of control vertices in the 2-ring surrounding the vertex that we are interested in. There are 4 types of vertices: $K$, $A$, $B$, $C$, as shown in the picture below. We are interested only in the submatrix of the subdivision matrix acting on the vector of control vertices consisting of $K, A_0, A_1, \ldots A_{n-1}$, where $n$ is the degree of the vertex in the center. Write the equation for computing $A_i^{j+1}$ and $K^{j+1}$ using $A_i^j$, $K^j$. Apply DFT to both sides of these equations.




The formulas relating the control points in the 2-ring on level $j+1$ to the control points on level $j$ in matrix form are written as

$$c^{j+1} = S'c^j$$

where $S'$ is our submatrix. When we apply DFT to our equations, this is equivalent to writing the matrix in a different basis. Write the general form of the subdivision matrix in the new basis. Observe that in the new basis the subdivision matrix consists of a $n \times n$ diagonal block, with a column vector of length $n$ attached on the left, a row vector of length $n$ on the top, and a single element in the upper left corner.

b. Find the eigenvalues and eigenvectors of the matrix in the transformed form. Recall that a left eigenvector $e$ with eigenvalue $\lambda$ is a row vector that satisfies

$eS' = \lambda e$. What are the left eigenvectors corresponding to the largest eigenvalues? Check that the eigenvalues satisfy the assumptions of Reif's condition. What else do we have to check to be able to assert smoothness of Loop's scheme?

e*. Invert the transformation that you have used to reduce the subdivision matrix to the simplified form and convert the right eigenvectors to the original basis. Recall that the right eigenvectors are used to compute the coefficients of a decomposition with respect to the basis of left eigenvectors. Write explicit formulas two tangent vectors of the subdivision surface at the central vertex in terms of the control points $K^0$, $A_0^0, \ldots A_{n-1}^0$.