Announcements

- Written project proposal due today.
- Second assignment available on Thursday.
- Access to CIS: see WWW.
- Thursday: light scattering. *HEPL*.
- Tuesday: rational curves (NURBS). *CIS*.

The second assignment will be a bit harder than the first one. It's still in the making. As you might have guessed, the programming part will have you write the drawing code of a B-spline drawing program — `scurvy`!

For those of you who need to use the advanced hardware in CIS (the RealityEngine or DASH), you may gain access to these resources by following the instructions posted on the WWW: see the class requirements page.

Note the different meeting place for this Thursday (and next Thursday).

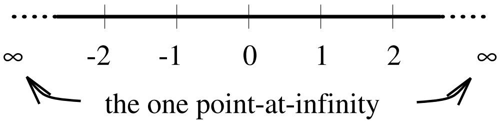
Are David Hsu and Afra Zomorodian here?

All color figures from the papers are on the WWW. I also brought the papers to class to see the pictures.

Today

Review & examples.

Projective Line

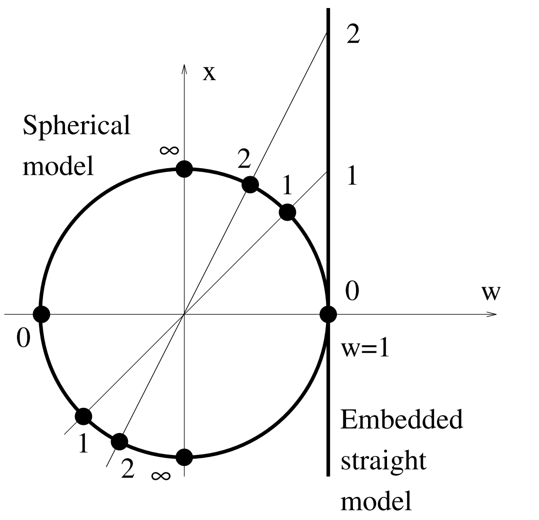


Straight model: $(p_w; p_x)$ is drawn at

- p_x/p_w if $p_w \neq 0$, or
- "far away at infinity" if $p_w = 0$.

As in the case of the plane, a point on the projective line is represented by pairs $(\lambda p_w; \lambda p_x)$, where $\lambda \neq 0$; if $p_w = 0$, it is the point-at-infinity (there is just one), otherwise it corresponds to the point p_x/p_w on the Euclidean line.

Projective Line



Spherical model: $(p_w; p_x)$ is drawn at

$$(p_w/l, p_x/l) \quad \text{and} \quad (-p_w/l, -p_x/l)$$

where $l = \sqrt{p_w^2 + p_x^2}$.

Although it should be called the "circular model", it's called "spherical" by analogy to the projective plane's spherical model.

The equations matching a projective point to its visualization have the form

$$\frac{(u, v)}{\sqrt{u^2 + v^2}}$$

which is clearly a point on the unit circle, as it is one unit of distance away from the origin.

In the drawing, you can see the correspondence between the two models. We first embed the straight model in the 2D plane at $w = 1$. And then, we project the embedded straight line onto the unit circle: this way, we project the straight model visualization of a projective point $\mathbf{p}$ onto the spherical model visualization of $\mathbf{p}$.

We can see why this projection scheme works by the following argument:

- If $p_w \neq 0$, then the projective point $(p_w; p_x)$ appears at $(1, p_x/p_w)$ on the embedded straight model.
- In order to project the point, we draw a straight line from the origin to the point and find its intersections with the unit circle. Such a straight line has slope p_x/p_w. The line joining the origin and $(p_w/l, p_x/l)$ and $(-p_w/l, -p_x/l)$ also has the same slope, and the last two points also lie on the unit circle, as we saw earlier. Hence, these points are the projections of the straight model visualization of $(p_w; p_x)$ onto the unit circle.
- If $p_w = 0$, then we can think of the projective point appearing far away, at infinity, on the embedded straight model. And hence, it projects to $(0, 1)$ and $(0, -1)$ on the unit circle. Alternatively, we can use the formulas to obtain

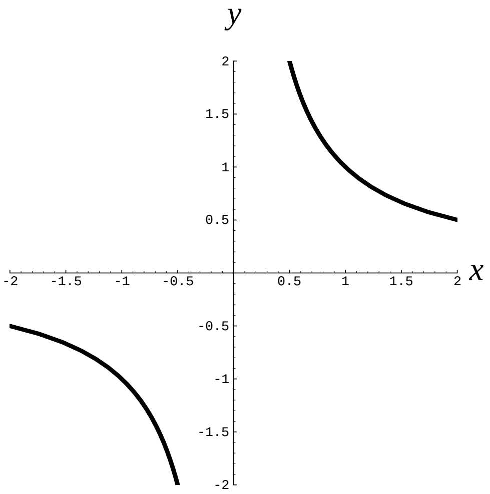
  $$(p_w/l, p_x/l) = (0, p_x/p_x) = (0, 1),$$

  and

  $$(-p_w/l, -p_x/l) = (0, -p_x/p_x) = (0, -1).$$

In a sense, the spherical model is better than the straight one because the point-at-infinity is actually visualized as a concrete, finite point on the wx plane. However, equidistant projective points are not drawn equally spaced on the unit circle, and thus the spherical model distorts distances.

Projective Plane

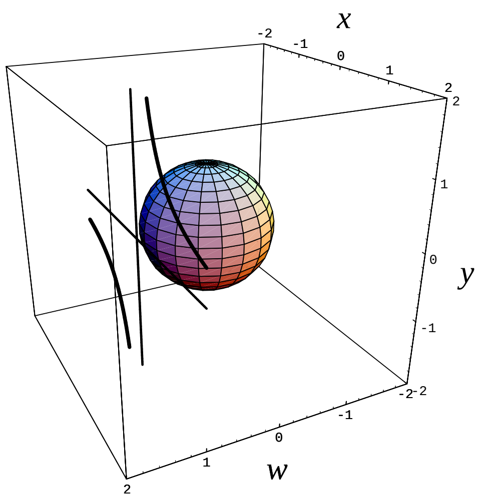


The projective hyperbola $p_x p_y = p_w^2$ drawn on the straight model.

This is just the homework problem. We'll see how we can draw the visualization of the projective curve $p_x p_y = p_w^2$ on the spherical model model *without* doing complex 3D calculations. We'll just extend the projection technique of the previous slide to the projective plane.

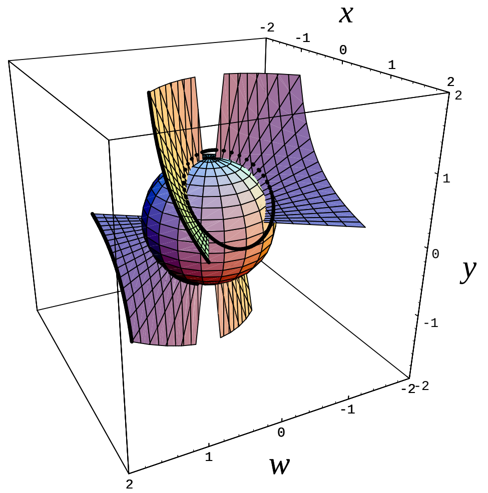
In this drawing, we are not attempting to draw the two points at infinity which lie on the hyperbola.

Projective Plane



Embedding the straight model in 3D space.

Projective Plane



Finding the spherical model visualization of the projective curve $p_x p_y = p_w^2$ by projection.

Polarization

Find the (3-)polar form of

$$\mathbf{p}(t) = (3t^3 + 9t, 6t^2 + 2).$$

Polarize each coordinate separately.

For $p_x(t) = 3t^3 + 9t$, we use the properties of polar forms in order:

Multiaffine: we can't raise t_i's to a power. That leaves us with only these possible coefficients: $\widehat{p}_x(t_1, t_2, t_3) =$

$$c_1 + c_2t_1 + c_3t_2 + c_4t_3 + c_5t_1t_2 + c_6t_2t_3 + c_7t_1t_3 + c_8t_1t_2t_3.$$

Symmetric: $\widehat{p}_x(t_1, t_2, t_3) = \widehat{p}_x(t_1, t_3, t_2)$ implies $c_3 = c_4$ and $c_5 = c_7$. Using all the symmetries (i.e all the permutations of the arguments), we conclude that

$$c_2 = c_3 = c_4 \quad \text{and} \quad c_5 = c_6 = c_7.$$

Diagonal: from the two previous steps, we deduce that

$$\widehat{p}_x(t, t, t) = c_1 + 3c_2t + 3c_5t^2 + c_8t^3.$$

In order to achieve $\widehat{p}_x(t, t, t) = p_x(t)$, we equate coefficients of identical powers of t to obtain: $c_1 = 0$, $c_2 = 3$ (and so $c_3 = c_4 = 3$), $c_5 = 0$ (and so $c_6 = c_7 = 0$), and $c_8 = 3$. Hence,

$$\widehat{p}_x(t_1, t_2, t_3) = 3t_1t_2t_3 + 3(t_1 + t_2 + t_3).$$

The first two steps are the same in deducing the polar form of $p_y(t) = 6t^2 + 2$. The last step requires that $c_1 = 2$, $c_2 = 0$ (and so $c_3 = c_4 = 0$), $c_5 = 2$ (and so $c_6 = c_7 = 2$), and $c_8 = 0$. Hence the polar form of $p_y(t)$ is

$$\widehat{p}_y(t_1, t_2, t_3) = 2(t_1t_2 + t_2t_3 + t_1t_3) + 2.$$

Multiaffine

The (3-)polar form of

$$\mathbf{p}(t) = (3t^3 + 9t, 6t^2 + 2)$$

is

$$\widehat{\mathbf{p}}(t_1, t_2, t_3) = (\widehat{p}_x, \widehat{p}_y),$$

where

$$\begin{aligned} \widehat{p}_x &= 3t_1t_2t_3 + 3(t_1 + t_2 + t_3) \\ \widehat{p}_y &= 2(t_1t_2 + t_2t_3 + t_1t_3) + 2. \end{aligned}$$

Show that when $t_2 = 2$, $t_3 = 3$, then $\widehat{\mathbf{p}}(t_1, t_2, t_3)$ varies affinely with t_1.

Just plug in the given values of t_2 and t_3 in the polar form to obtain

$$\begin{aligned} \widehat{\mathbf{p}}(t_1, 2, 3) &= (18t_1 + 3(t_1 + 5), 2(2t_1 + 6 + 3t_1) + 2) \\ &= (21t_1 + 15, 10t_1 + 14). \end{aligned}$$

Multiaffine

Let $f(x) = ax + b$ for unknown constants a, and b. If

$$f(c) = f_c \quad \text{and} \quad f(d) = f_d,$$

for $c \neq d$, find $f(t)$ for arbitrary t.

One way to solve this problem is finding the unknowns a and b by solving the following system of equations:

$$\begin{aligned} ac + b &= f_c \\ ad + b &= f_d. \end{aligned}$$

Another way, the one we use during the de Casteljau algorithm, is by expressing t as a barycentric combination of c and d,

$$t = wc + (1 - w)d,$$

where w is found by rearranging the terms of the equation

$$w = \frac{t - d}{c - d}.$$

Then

$$\begin{aligned} f(t) &= f(wc + (1 - w)d) \\ &= a(wc + (1 - w)d) + b \\ &= awc + ad - awd + b \\ &= awc + ad - awd + b + wb - wb \\ &= w(ac + b) + (1 - w)(ad + b) \\ &= wf_c + (1 - w)f_d. \end{aligned}$$

Bézier Points

Find the (canonical) Bézier points $\mathbf{b}_1, \ldots, \mathbf{b}_4$ of

$$\mathbf{p}(t) = (3t^3 + 9t, 6t^2 + 2).$$

By "canonical" we mean that the curve passes through the first Bézier point at $t = 0$ and through the last one at $t = 1$.

One way to do this is to use the fact that:

$$\begin{aligned} \mathbf{p}(0) &= \mathbf{b}_1 \\ \mathbf{p}(1) &= \mathbf{b}_4 \\ \mathbf{p}'(0) &= 3(\mathbf{b}_2 - \mathbf{b}_1) \\ \mathbf{p}'(1) &= 3(\mathbf{b}_4 - \mathbf{b}_3). \end{aligned}$$

In this case, we find the left hand-sides by evaluation and differentiation and solve four equations in four unknowns to find the Bézier points. (Actually, we have to solve two sets of four scalar equations in four unknowns, since the equations above are equivalent to two scalar equations each — one for each of the x and y components).

An easier way to find the Bézier points is to polarize $\mathbf{p}(t)$. Then, the Bézier points are trivially given by

$$\begin{aligned} \mathbf{b}_1 &= \hat{\mathbf{p}}(0, 0, 0) \\ \mathbf{b}_2 &= \hat{\mathbf{p}}(0, 0, 1) \\ \mathbf{b}_3 &= \hat{\mathbf{p}}(0, 1, 1) \\ \mathbf{b}_4 &= \hat{\mathbf{p}}(1, 1, 1). \end{aligned}$$

We found the polar form of the given curve earlier; evaluating it for the polar arguments above yields:

$$\begin{aligned} \mathbf{b}_1 &= (0, 2) \\ \mathbf{b}_2 &= (3, 2) \\ \mathbf{b}_3 &= (6, 4) \\ \mathbf{b}_4 &= (12, 8). \end{aligned}$$

Bernstein Polynomials

Derive the (canonical) Bernstein polynomials for cubic curves.

By "canonical" we mean that these Bernstein polynomials form the basis functions for canonical Bézier points.

Recall how the de Casteljau algorithm finds the curve point $\mathbf{p}(t)$ given the Bézier points $\mathbf{b}_1, \ldots, \mathbf{b}_4$:

$$\begin{aligned}
\mathbf{b}_1^1 &= (1-t)\mathbf{b}_1 + t\mathbf{b}_2 \\
\mathbf{b}_2^1 &= (1-t)\mathbf{b}_2 + t\mathbf{b}_3 \\
\mathbf{b}_3^1 &= (1-t)\mathbf{b}_3 + t\mathbf{b}_4 \\
\mathbf{b}_1^2 &= (1-t)\mathbf{b}_1^1 + t\mathbf{b}_2^1 \\
\mathbf{b}_2^2 &= (1-t)\mathbf{b}_2^1 + t\mathbf{b}_3^1 \\
\mathbf{p}(t) &= (1-t)\mathbf{b}_1^2 + t\mathbf{b}_2^2.
\end{aligned}$$

Eliminating the intermediate points $\mathbf{b}_i^j$, we obtain

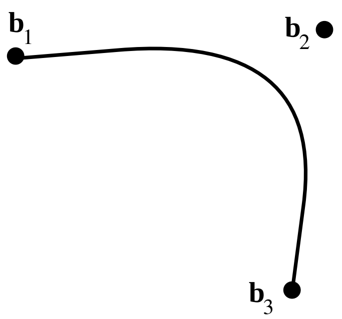
$$\mathbf{p}(t) = \mathbf{b}_1 B_1(t) + \mathbf{b}_2 B_2(t) + \mathbf{b}_3 B_3(t) + \mathbf{b}_4 B_4(t)$$

where

$$\begin{aligned}
B_1(t) &= (1-t)^3 \\
B_2(t) &= 3(1-t)^2 t \\
B_3(t) &= 3(1-t)t^2 \\
B_4(t) &= t^3.
\end{aligned}$$

Degree Raising

Find the (canonical) Bézier points of the quadratic curve shown below, viewed as a (degenerate) cubic. You are given the (canonical) Bézier points of the curve viewed as a quadratic.



The key is to use polar forms. The given Bézier points fully define the 2-polar form of the curve $\mathbf{p}(t)$. The 3-polar form of $\mathbf{p}(t)$ can be easily found from the 2-polar form:

$$\widehat{\mathbf{p}}(t_1, t_2, t_3) = \frac{1}{3}[\widehat{\mathbf{p}}(t_1, t_2) + \widehat{\mathbf{p}}(t_1, t_3) + \widehat{\mathbf{p}}(t_2, t_3)].$$

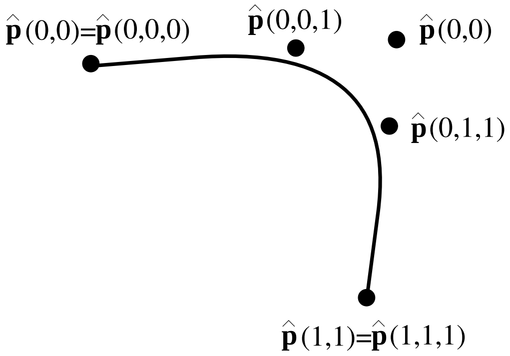
This works because the right-hand side defines a multiaffine, symmetric function such that $\widehat{\mathbf{p}}(t,t,t) = \mathbf{p}(t)$:

- It is multiaffine because the 2-polar form is multiaffine and the right-hand side contains simple sums of 2-polar forms.
- It is symmetric since the 2-polar form is symmetric and all possible ways to selected two of the three polar arguments appear on the right-hand side.
- $\widehat{\mathbf{p}}(t,t,t) = \mathbf{p}(t)$ since $\widehat{\mathbf{p}}(t,t) = \mathbf{p}(t)$.

Using this identity, we can find the Bézier points as follows:

$$\begin{aligned}
\widehat{\mathbf{p}}(0,0,0) &= \widehat{\mathbf{p}}(0,0) \\
\widehat{\mathbf{p}}(0,0,1) &= \frac{1}{3}[\widehat{\mathbf{p}}(0,0) + 2\widehat{\mathbf{p}}(0,1)] \\
\widehat{\mathbf{p}}(0,1,1) &= \frac{1}{3}[\widehat{\mathbf{p}}(1,1) + 2\widehat{\mathbf{p}}(0,1)] \\
\widehat{\mathbf{p}}(1,1,1) &= \widehat{\mathbf{p}}(1,1).
\end{aligned}$$

Graphically,



Homogenization

Homogenize both range and domain of

$$\mathbf{p}(t) = (3t^3 + 9t, 6t^2 + 2).$$

Homogenization turns a map from the Euclidean line to the Euclidean plane into one from the projective line to the projective plane. So, the homogenized $\mathbf{p}(t)$ maps a projective point $(t_w; t_x)$ to a projective point $(p_w; p_x, p_y)$. Homogenizing the domain yields

$$\mathbf{p}[(t_w; t_x)] = (3\left(\frac{t_x}{t_w}\right)^3 + 9\frac{t_x}{t_w}, 6\left(\frac{t_x}{t_w}\right)^2 + 2),$$

and homogenizing the range yields

$$\begin{aligned} \mathbf{p}[(t_w; t_x)] &= (1; 3\left(\frac{t_x}{t_w}\right)^3 + 9\frac{t_x}{t_w}, 6\left(\frac{t_x}{t_w}\right)^2 + 2) \\ &= (t_w^3; 3t_x^3 + 9t_x t_w^2, 6t_x^2 t_w + 2t_w^3). \end{aligned}$$

Homogenization

Homogenize both range and domain of

$$\hat{\mathbf{p}}(t, r, s) = (3trs + 3(t + r + s), 2(tr + rs + ts) + 2).$$

Proceeding just as in the previous problem, we first homogenize the domain to obtain

$$\hat{\mathbf{p}}[(t_w;t_x),(r_w;r_x),(s_w;s_x)] = (\hat{p}_x, \hat{p}_y)$$

where

$$\begin{aligned}\hat{p}_x &= 3\frac{t_x}{t_w}\frac{r_x}{r_w}\frac{s_x}{s_w} + 3\left(\frac{t_x}{t_w}+\frac{r_x}{r_w}+\frac{s_x}{s_w}\right)\\ \hat{p}_y &= 2\left(\frac{t_x}{t_w}\frac{r_x}{r_w}+\frac{r_x}{r_w}\frac{s_x}{s_w}+\frac{t_x}{t_w}\frac{s_x}{s_w}\right)+2.\end{aligned}$$

Homogenizing the range too produces

$$\hat{\mathbf{p}}[(t_w;t_x),(r_w;r_x),(s_w;s_x)] = (\hat{p}_w;\hat{p}_x, \hat{p}_y)$$

where

$$\begin{aligned}\hat{p}_w &= t_w r_w s_w\\ \hat{p}_x &= 3t_x r_x s_x + 3(t_x r_w s_w + r_x t_w s_w + s_x t_w r_w)\\ \hat{p}_y &= 2(t_x r_x s_w + r_x s_x t_w + t_x s_x r_w) + 2t_w r_w s_w.\end{aligned}$$

Observe that we were told to homogenize the polar form of the cubic curve on the previous example. Hence, it's trivial to show that

$$\hat{\mathbf{p}}[(t_w;t_x),(t_w;t_x),(t_w;t_x)] = \mathbf{p}[(t_w;t_x)].$$

Differentiation

Find $\mathbf{p}'(1)$ given the polar form of $\mathbf{p}(t)$:

$$\hat{\mathbf{p}}(t,r,s) = (3trs + 3(t+r+s), 2(tr+rs+ts)+2).$$

A simple way to do this is by finding $\mathbf{p}(t) = \hat{\mathbf{p}}(t,t,t)$ and then differentiating. This yields

$$\mathbf{p}(t) = (3t^3 + 9t, 6t^2 + 2)$$

and so

$$\mathbf{p}'(t) = (9t^2 + 9, 12t + 2)$$

and

$$\mathbf{p}'(1) = (18, 14).$$

A harder way to do this, but more instructive, is to use homogenized polar forms:

$$\begin{aligned}\mathbf{p}'(1) &= \lim_{h\to 0}\frac{\mathbf{p}(1+h)-\mathbf{p}(1)}{h}\\ &= \lim_{h\to 0}\frac{\hat{\mathbf{p}}(1+h,1+h,1+h)-\hat{\mathbf{p}}(1,1,1)}{h}.\end{aligned}$$

Using the result of the previous slide, i.e.

$$\hat{\mathbf{p}}[(t_w;t_x),(r_w;r_x),(s_w;s_x)] = (\hat{p}_w;\hat{p}_x, \hat{p}_y)$$

where

$$\begin{aligned}\hat{p}_w &= t_w r_w s_w\\ \hat{p}_x &= 3t_x r_x s_x + 3(t_x r_w s_w + r_x t_w s_w + s_x t_w r_w)\\ \hat{p}_y &= 2(t_x r_x s_w + r_x s_x t_w + t_x s_x r_w) + 2t_w r_w s_w,\end{aligned}$$

we conclude that

$$\hat{\mathbf{p}}[(1;1+h),(r_w;r_x),(s_w;s_x)] = (\hat{p}_w;\hat{p}_x, \hat{p}_y)$$

where

$$\begin{aligned}\hat{p}_w &= r_w s_w\\ &\quad +h0\\ \hat{p}_x &= 3r_x s_x + 3(r_w s_w + r_x s_w + s_x r_w)\\ &\quad +h[3r_x s_x + 3r_w s_w]\\ \hat{p}_y &= 2(r_x s_w + r_x s_x + s_x r_w) + 2r_w s_w\\ &\quad +h[2(r_x s_w + s_x r_w)].\end{aligned}$$

This can be written simply as

$$\widehat{\mathbf{p}}[(1;1),(r_w;r_x),(s_w;s_x)] + h\widehat{\mathbf{p}}[\delta,(r_w;r_x),(s_w;s_x)]$$

where the first summand absorbs the first row for each component and the second summand absorbs the second row.

As we pointed out last time, in order to add these quantities pairwise, we have to abandon our understanding of arguments and results as projective points. For example, the projective points (2; 4) and (1; 3), corresponding to the Euclidean points 2 and 3, do not add up to 5 by pairwise addition; instead, we get (3; 7) whose corresponding Euclidean point is 7/3.

The proper interpretation for both arguments and results is that of sites, with permissible weights 0 or 1 alone. When the weight is 0, the multiples of the remaining components of the pair or triple are *not* identified. For example, (0; 7) and (0; 1) define the two *distinct* one dimensional vectors 7 and 1.

By repeatedly applying the identity

$$\widehat{\mathbf{p}}(1+h,\cdot,\cdot) = \widehat{\mathbf{p}}(1,\cdot,\cdot) + h\widehat{\mathbf{p}}(\delta,\cdot,\cdot)$$

to $\widehat{\mathbf{p}}(1+h,1+h,1+h)$, we conclude that

$$\begin{aligned}
&\widehat{\mathbf{p}}(1+h,1+h,1+h)\\
&= \widehat{\mathbf{p}}(1,1+h,1+h) + h\widehat{\mathbf{p}}(\delta,1+h,1+h)\\
&= \widehat{\mathbf{p}}(1,1,1+h) + h\widehat{\mathbf{p}}(1,\delta,1+h) + h\widehat{\mathbf{p}}(\delta,1,1+h) +\\
&\quad +h^2\widehat{\mathbf{p}}(\delta,\delta,1+h)\\
&= \widehat{\mathbf{p}}(1,1,1) + h\widehat{\mathbf{p}}(1,1,\delta) + 2h\widehat{\mathbf{p}}(1,\delta,1+h)\\
&\quad +h^2\widehat{\mathbf{p}}(\delta,\delta,1) + h^3\widehat{\mathbf{p}}(\delta,\delta,\delta)\\
&= \widehat{\mathbf{p}}(1,1,1) + 3h\widehat{\mathbf{p}}(1,1,\delta) + 3h^2\widehat{\mathbf{p}}(1,\delta,\delta) + h^3\widehat{\mathbf{p}}(\delta,\delta,\delta).
\end{aligned}$$

So, finally,

$$\begin{aligned}
\mathbf{p}'(1) &= \lim_{h\to 0}\frac{\widehat{\mathbf{p}}(1+h,1+h,1+h) - \mathbf{p}(1,1,1)}{h}\\
&= \lim_{h\to 0}[3\widehat{\mathbf{p}}(1,1,\delta) + 3h\widehat{\mathbf{p}}(1,\delta,\delta) + h^2\widehat{\mathbf{p}}(\delta,\delta,\delta)]\\
&= 3\widehat{\mathbf{p}}(1,1,\delta)\\
&= (0;18,14).
\end{aligned}$$

Parametric Continuity

Prove that C^1 continuity is guaranteed at $t = 1$ between the two adjacent curves $\mathbf{p}(t)$ and $\mathbf{q}(t)$ iff

$$\widehat{\mathbf{p}}(1,1,s) = \widehat{\mathbf{q}}(1,1,s)$$

for all s.

Since C^1 continuity necessitates C^0, the equality must hold at $s = 1$ in order for $\mathbf{p}(1)$ and $\mathbf{q}(1)$ to coincide. What about $s \neq 1$?

We saw earlier that

$$\begin{aligned}
\mathbf{p}'(1) &= 3\widehat{\mathbf{p}}(1,1,\delta)\\
\mathbf{q}'(1) &= 3\widehat{\mathbf{q}}(1,1,\delta).
\end{aligned}$$

So equality of the derivatives is equivalent to

$$\widehat{\mathbf{p}}(1,1,\delta) = \widehat{\mathbf{q}}(1,1,\delta).$$

From the earlier identities

$$\begin{aligned}
\widehat{\mathbf{p}}(1+h,\cdot,\cdot) &= \widehat{\mathbf{p}}(1,\cdot,\cdot) + h\widehat{\mathbf{p}}(\delta,\cdot,\cdot)\\
\widehat{\mathbf{q}}(1+h,\cdot,\cdot) &= \widehat{\mathbf{q}}(1,\cdot,\cdot) + h\widehat{\mathbf{q}}(\delta,\cdot,\cdot),
\end{aligned}$$

which hold for all h, it follows that equality of the derivatives is also equivalent to

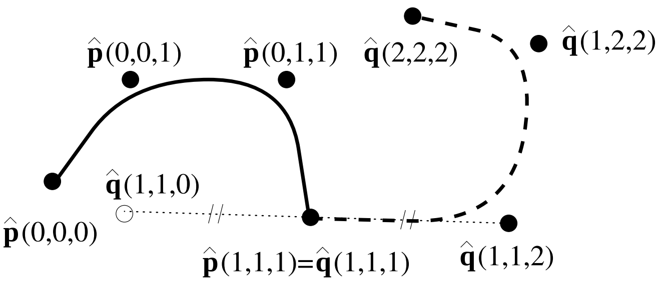
$$\widehat{\mathbf{p}}(1+h,1,1) - \widehat{\mathbf{p}}(1,1,1) = \widehat{\mathbf{q}}(1+h,1,1) - \widehat{\mathbf{q}}(1,1,1)$$

which, using C^0 continuity and substituting $s = 1+h$, reduces to

$$\widehat{\mathbf{p}}(s,1,1) = \widehat{\mathbf{q}}(s,1,1).$$

Parametric Continuity

C^0 but not C^1:

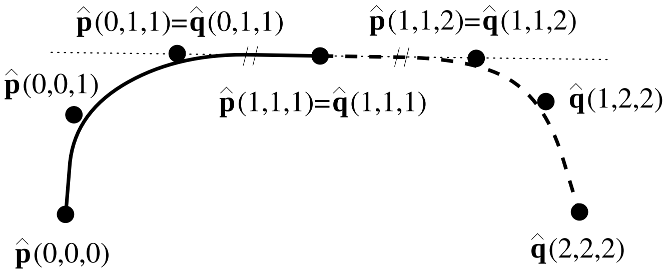


Can't be C^1 since

$$\hat{\mathrm{p}}(0,1,1) \neq \hat{\mathrm{q}}(0,1,1).$$

Parametric Continuity

(C^0 and) C^1:



If

$$\hat{\mathrm{p}}(s,1,1) = \hat{\mathrm{q}}(s,1,1)$$

is true for two distinct values of s, say s_1 and s_2 (0 and 1 on the example), then it's true for all values of s. This follows from the multiaffine-ness of the polar forms: $\hat{\mathrm{p}}(s,1,1)$ can be expressed as

$$w\hat{\mathrm{p}}(s_1,1,1) + (1-w)\hat{\mathrm{p}}(s_2,1,1)$$

where w depends on s_1 and s_2 only. The same is true for $\hat{\mathrm{q}}(s,1,1)$, i.e.

$$\hat{\mathrm{q}}(s,1,1) = w\hat{\mathrm{q}}(s_1,1,1) + (1-w)\hat{\mathrm{q}}(s_2,1,1)$$

and so,

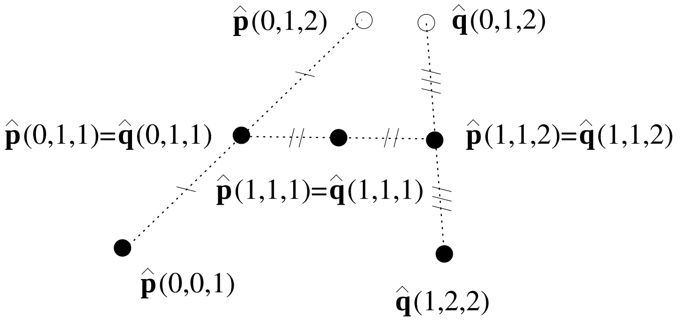
$$\hat{\mathrm{p}}(s,1,1) = \hat{\mathrm{q}}(s,1,1)$$

holds for all s.

The reason that "C^0 and" is placed in parentheses is that C^1 continuity necessitates C^0. Hence, the parenthetical observation is, strictly speaking, redundant.

Parametric Continuity

C^1 but not C^2:



For C^2,

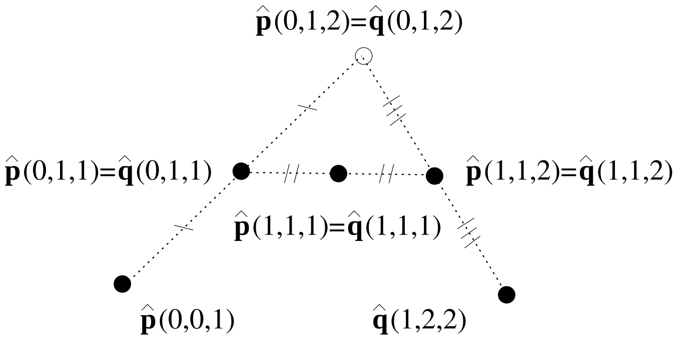
$$\hat{\mathrm{p}}(s_1,s_2,1) = \hat{\mathrm{q}}(s_1,s_2,1)$$

must hold for all s_1, s_2. Here violated for $s_1 = 0$ and $s_2 = 2$.

The condition for C^2 continuity is proven much the same way as the condition for C^1 continuity.

Parametric Continuity

(C^1 and) C^2:



The dotted line forms the *A-frame*.

By a similar argument to the one used for C^1 continuity, it can be shown that if

$$\hat{\mathbf{p}}(s_1, s_2, 1) = \hat{\mathbf{q}}(s_1, s_2, 1)$$

is true for the shown three pairs of values for s_1 and s_2 (namely, 0 and 1, 0 and 2, and 1 and 2), then it's true for all values of s_1 and s_2.

B-Splines

Given

Degree of segments.

Knots: when (values of t) segments start and end.

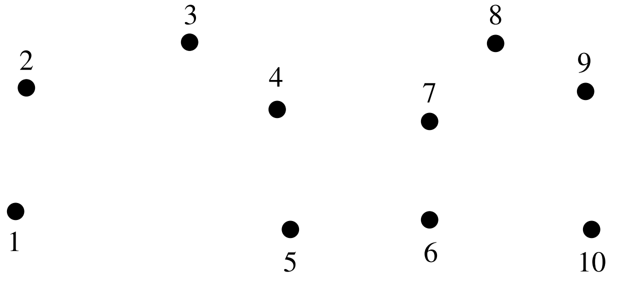
Knot multiplicities: desired continuity between adjacent segments.

de Boor points defining shape of each segment.

draw the spline.

B-Splines: Example

Quadratic B-spline with *knots* 0, 1, 2, 3, 4, 5. Knots 1, 2, 3, and 4 have *multiplicities* 1, 2, 1, and 3 respectively. The de Boor points are:



We do not need to know the multiplicities of the first and last knot since there is no segment before the first one, and no segment after the last one. For reasons that will become clear soon, we choose the multiplicity of these knots to be the same as the degree of the spline.

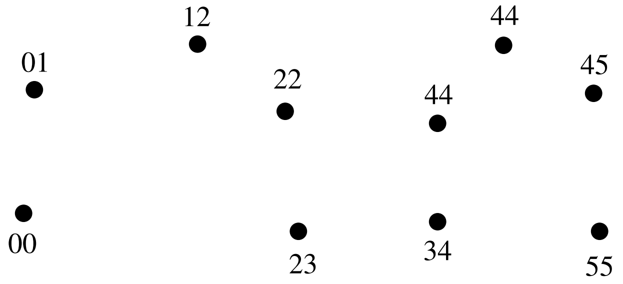
Also, how many de Boor points are needed? We'll address that issue soon.

B-Splines: Drawing

Step 1: Write down *knot sequence*, i.e. knots repeated according to their multiplicities.

0, 0, 1, 2, 2, 3, 4, 4, 4, 5, 5

Step 2: label de Boor points with as many adjacent knot sequence elements as the degree of the spline:



So how many de Boor points did we need?

$$1 + \sum_{i=2}^{i=n-1} m_i + d$$

where d is the degree of the spline, n is the number of knots, and m_i are the knot multiplicities. The sum is taken over all the knots except the first and the last one.

B-Splines: Drawing

Step 3: assign de Boor points to each segment:

Segment	00	1	22	3	444	55
$0 \ldots 1$	00 0	1 1	2			
$1 \ldots 2$	0	1 1	2 22			
$2 \ldots 3$			22 2	3 3	4	
$3 \ldots 4$			2	3 3	4 44	
$4 \ldots 5$					44 4	5 55

How do we do the assignment? In two steps:

1. Every segment $a \ldots b$ is assigned the de Boor point labeled ab.

2. The same segment also gets the de Boor points before and after the one labeled ab.

For higher degree splines, if the degree is even, we start assignments with the de Boor point which has ab in the *middle* of its label; for odd degrees, we start assignments with the *two* de Boor points whose labels have ab as near the middle as possible. For both odd and even degrees d, we then keep assigning de Boor points before and after the selected ones interchangeably, until we collect the $d+1$ de Boor points needed to define a degree d polynomial curve.

For this method to work for the first and last segment, we need to assign a multiplicity of exactly d to the first and last knot. That's why we only required the user to specify the multiplicities of the *interior* knots.

Why do we do the assignments this way? In order to get the desired continuity. To show this, let's consider an example: from the table we can see that segments $0 \ldots 1$ and $1 \ldots 2$ share two de Boor points (labeled 01 and 12). Using $\mathbf{p}_0(t)$ and $\mathbf{p}_1(t)$ to denote the polynomial curves corresponding respectively to these segments, we conclude that

$$\begin{aligned} \hat{\mathbf{p}}_0(0,1) &= \hat{\mathbf{p}}_1(0,1) \\ \hat{\mathbf{p}}_0(2,1) &= \hat{\mathbf{p}}_1(2,1), \end{aligned}$$

and so $\hat{\mathbf{p}}_0(s,1) = \hat{\mathbf{p}}_1(s,1)$ for all s. Hence the segments $0 \ldots 1$ and $1 \ldots 2$ meet with C^1 continuity at $t=1$. Segments $1 \ldots 2$ and $2 \ldots 3$ share only one de Boor point (labeled 22) and thus they do not necessarily meet with C^1 continuity at $t=2$; but C^0 is guaranteed. And so on...

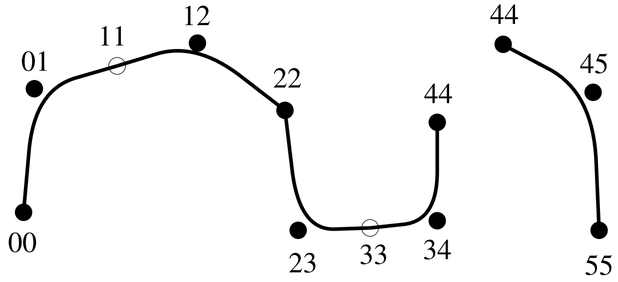
Repeating knots "push apart" the vertical lines in this table and thus make adjacent segments share fewer de Boor points and thus have lesser (guaranteed) continuity. That's why the higher a knot's multiplicity, the more derivatives may be broken across the two segments which meet at the knot. Of course, the positions of the de Boor points might be set in such a way that no derivatives are broken at all — that's why repeating knots gives us *permission* to break a derivative; we have *no obligation* to break it.

This intuition behind multiplicities also explains why a knot's multiplicity can be an most $d+1$ where d is the spline's degree. If a knot if repeated $d+1$ times, adjacent segments share no de Boor points at all, and so are fully independent of each other. More repetitions than $d+1$ would mean that the user would have to specify more de Boor points (for the added labels in the knot sequence) — points which we would never use to draw the spline!

Finally, notice the *local control* property: the de Boor point labeled 23 affects only segments $2 \ldots 3$ and $3 \ldots 4$, without affecting $0 \ldots 1$. This is extremely desirable for modeling, since carefully tweaking the shape of a car's hood should not turn a sedan into a hatchback.

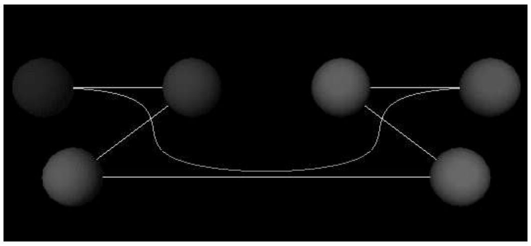
B-Splines: Drawing

Step 4: draw segments using the de Boor algorithm:

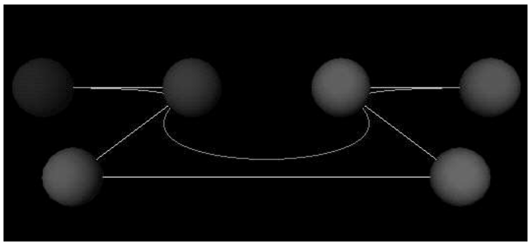


B-Splines: Nonuniform Knots

Uniform: knots 0, 3, 6, 9.



Nonuniform: knots 0, 1, 8, 9.



The intuition behind the shape change is the following: in the uniform case, a particle moving along the curve can spend equal times traveling along each segment. By forcing the particle to spend very little time traveling along the first and third segment, the segment has to become as short as its de Boor points allow it to be: as a straight line is the fastest way to get from one point to another, the particle assumes an almost straight path between adjacent de Boor points.

Non-uniform knot spacing is not frequently used in commercial modelers for tweaking the shape of a curve. Rather, its main application is in animation, where the movement of the camera is usually defined by a spline: the de Boor points are used to specify the basic geometry of the path, and knot spacing is subsequently tweaked to get the right timing — usually, a movement with constant speed. Of course, this tweaking does change the path itself, but not too much, in practice.

Reference

`scurvy`: B-spline curve visualization and manipulation software.

Try out the sample parameter files.

See the WWW.

`scurvy` is a software package I wrote to understand B-spline curves. Its features include visualization of arbitrary B-spline curves, interactive modification of the de Boor point positions and knot spacing, visualization of the interpolations performed by the de Boor algorithm, and animation of a particle moving on the curve. Play with it to gain some intuition about B-splines.

The WWW includes `scurvy`'s documentation, the executable, and descriptions of the sample parameter files.