Announcements

- First assignment due Thursday, 2:30p.
- Project proposal meetings.
- Course reader.
- Thursday: PDI field trip.
- Tuesday: Lyle Ramshaw. *CIS*.

The assignment is due on Thursday, right before the PDI field trip. Start now, if you haven't yet. It's not long, but it requires more work than what you can fit on the back of an envelope.

Sign up by Friday for a project proposal meeting with me. The signup sheet is electronic, and is available on the WWW. Meetings will take place next week. You are expected to come to the meetings with a preliminary form of the project proposal in your head. Your written proposals are due Tue 10/31. See the WWW for a description of what a project proposal should contain.

Extra copies of the course reader are available for auditors.

On Thursday, we are going on a field trip to PDI. Who has a car?

Lyle will lecture next week on splines.

Today

- Polar forms of polynomials.
- Polarization.
- de Casteljau revisited.
- de Boor points and algorithm.
- Homogenization of curves.
- Differentiation of polar forms.

Polar Forms: Introduction

A polynomial $f(t)$ has degree d iff t's largest exponent *doesn't exceed* d. Let $n = d + 1$.

Ways to write a polynomial $f(t)$:

Monomial form: $f(t) = a_1 + a_2 t + \cdots + a_n t^d$.

Arbitrary basis: $f(t) = b_1 B_1(t) + \cdots + b_n B_n(t)$.

Polar form: *Multiaffine*, *symmetric* function with d arguments $\widehat{f}(t_1, \ldots, t_d)$ such that

$$\widehat{f}(\underbrace{t, \ldots, t}_{d \text{ terms}}) = f(t).$$

Some authors refuse to call a quadratic a cubic. We will not worry about that. However, if we know that a cubic is really a quadratic, we'll call the cubic a *degenerate* one.

We will explain what "multiaffine" and "symmetric" mean later on. First, observe that the number of arguments in a polar form depends on the degree d. So...

Polar Forms: Degree

A particular $f(t)$ has many *different* polar forms, depending on what we consider the degree of $f(t)$ to be. e.g.

$$f(t) = t^2 + 3$$

can be seen as a

Quadratic: polar form $\widehat{f}(t_1, t_2)$ has 2 arguments.

Cubic: polar form $\widehat{f}(t_1, t_2, t_3)$ has 3 arguments.

Usually, can tell degree d from context. If not, we say *the d-polar form* of $f(t)$.

Polar Forms: Nomenclature

Theory emerged in three different research projects: de Casteljau, de Boor, Ramshaw. So, many names and notations:

Polar form or Blossom: we use $\widehat{f}(t_1, \ldots, t_d)$ for polar form of $f(t)$. Others use $f(t_1, \ldots, t_d)$ for polar form of $F(t)$.

Polarization or Blossoming: finding $\widehat{f}(t_1, \ldots, t_d)$ given $f(t)$.

$f(t)$ is called *the univariate form*.

To avoid proliferating subscripts, we will now assume cubics ($d = 3$), unless we state otherwise. As usual, our definitions can be extended trivially to higher degree polynomials.

Ramshaw suggests

- "Polar form" because it implies "same entities, different representation".
- "Blossoming" because it implies "revelation of hidden structure".

Polar Forms: Symmetric

A *symmetric* function is one where, for all t_1, t_2, t_3,

$$\hat{f}(t_1, t_2, t_3) = \hat{f}(s_1, s_2, s_3)$$

where s_1, s_2, s_3 is any *permutation* of t_1, t_2, t_3 (i.e. same arguments in different order), i.e.

$$\begin{aligned} \hat{f}(t_1, t_2, t_3) &= \hat{f}(t_1, t_3, t_2) \\ &= \hat{f}(t_2, t_3, t_1) \\ &= \hat{f}(t_2, t_1, t_3) \\ &= \hat{f}(t_3, t_1, t_2) \\ &= \hat{f}(t_3, t_2, t_1). \end{aligned}$$

We have assumed a function of 3 arguments. Substitute d for 3 and you get the definition of a symmetric function of d arguments. In short, a symmetric function is one whose arguments can be written in any order without changing the output.

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Polar Forms: Multiaffine

A *multiaffine* function is one where there exist constants a and b such that

$$\hat{f}(t_1, t_2, t_3) = at_i + b$$

when *only* t_i changes with the rest of the t_j's ($i \neq j$) *held fixed.* a and b depend only on the particular, fixed, values of the t_j's.

Example: fixing $t_2 = 1$ and $t_3 = 3$,

$$\hat{f}(t, 1, 3) = \frac{4}{3}t + 4.$$

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Again, we have assumed a function of 3 arguments. Substitute d for 3 and you get the definition of a multiaffine function of d arguments.

Because of symmetry, the constants a and b do not depend on i, i.e. they are insensitive to which one of the t_i's changes. In the example, I can write $\hat{f}(t, 1, 3)$ as $\hat{f}(1, t, 3)$ by symmetry, and now it's the second argument that changes; still, however,

$$\hat{f}(1, t, 3) = \hat{f}(t, 1, 3) = \frac{4}{3}t + 4,$$

and thus a is still 4/3 and $b = 4$.

In general, an arbitrary function $f(\ldots, t, \ldots)$ is

linear with respect to ("w.r.t.") one of its arguments t if it is a scalar multiple of t, i.e. $f(\ldots, t, \ldots) = at$.

affine w.r.t. one of its arguments t if $f(\ldots, t, \ldots) = at + b$.

multilinear if it's linear w.r.t. all of its arguments.

multiaffine if it's affine w.r.t. all of its arguments.

where a and b above are constants.

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Polar Forms: Diagonal

$$\widehat{f}(t, t, t) = f(t).$$

A value of $\widehat{f}(t_1, t_2, t_3)$ is a *polar value* of $f(t)$. Each of the t_i's is a *polar argument*.

A polar value where $t_1 = t_2 = t_3 = t$ is a *diagonal value* of $f(t)$. t is a *diagonal argument*.

The two properties of the polar form $\widehat{f}(t_1, t_2, t_3)$ of $f(t)$ that we have presented so far, symmetry and multiaffineness have nothing to do with $f(t)$. This last property relates the polar and univariate forms.

Polarization: Example

Example: let $f(t) = t^3$; find $\widehat{f}(t_1, t_2, t_3)$.

Multiaffine: can't raise t_i's to a power; list all possible coefficients: $\widehat{f}(t_1, t_2, t_3) =$

$$\begin{array}{ccccccc} c_1 & + & c_2 t_1 & + & c_3 t_2 & + & c_4 t_3 \quad + \\ c_5 t_1 t_2 & + & c_6 t_2 t_3 & + & c_7 t_1 t_3 & + & c_8 t_1 t_2 t_3. \end{array}$$

Symmetric: $\widehat{f}(t_1, t_2, t_3) = \widehat{f}(t_1, t_3, t_2)$ implies $c_3 = c_4$ and $c_5 = c_7$. With all symmetries,

$$c_2 = c_3 = c_4 \quad \text{and} \quad c_5 = c_6 = c_7.$$

Diagonal: from above,

$$\widehat{f}(t, t, t) = c_1 + 3c_2 t + 3c_5 t^2 + c_8 t^3.$$

To get $\widehat{f}(t, t, t) = f(t)$, set all c_i's to zero, except $c_8 = 1$:

$$\widehat{f}(t_1, t_2, t_3) = t_1 t_2 t_3.$$

We never had to make choices in finding the polar form. That is, we never had any alternatives for selecting $\widehat{f}(t_1, t_2, t_3)$. This suggests that the polar form of a given polynomial (assuming a particular degree, i.e. the d-polar form) is unique. Also, for a given polar form $\widehat{f}(t_1, t_2, t_3)$, there is only one polynomial $f(t)$, given by $\widehat{f}(t, t, t)$. So, there is a one-to-one correspondence between the two entities $f(t)$ and $\widehat{f}(t, t, t)$, and hence they are just two different *forms* of the same underlying entity. That's why Ramshaw likes the term *polar form*: a *polar form* is just another way of writing a polynomial.

Polarization: General Case

First two steps, using multiaffine and symmetric character of polar form, are the same:

$$\begin{aligned}\hat{f}(t_1,t_2,t_3) &= c_1 + c_2(t_1+t_2+t_3) \\ &+ c_5(t_1t_2+t_2t_3+t_1t_3) + c_8t_1t_2t_3.\end{aligned}$$

Diagonal is

$$\hat{f}(t,t,t) = c_1 + 3c_2t + 3c_5t^2 + c_8t^3.$$

So, since $\hat{f}(t,t,t) = f(t)$, polar form of

$$f(t) = a_1 + a_2t + a_3t^2 + a_4t^3$$

is

$$\begin{aligned}\hat{f}(t_1,t_2,t_3) &= a_1 + \frac{a_2}{3}(t_1+t_2+t_3) \\ &+ \frac{a_3}{3}(t_1t_2+t_2t_3+t_1t_3) + a_4t_1t_2t_3.\end{aligned}$$

Polarization: Degree Raising

Assume $\hat{f}(t_1,t_2,t_3)$ is the 3-polar form of $f(t)$. What is the 4-polar form?

$$\begin{aligned}\hat{f}(t_1,t_2,t_3,t_4) = \frac{1}{4}\ [\ & \hat{f}(t_1,t_2,t_3) + \\ & \hat{f}(t_1,t_2,t_4) + \\ & \hat{f}(t_1,t_3,t_4) + \\ & \hat{f}(t_2,t_3,t_4)]\end{aligned}$$

because it is multiaffine, symmetric, $\hat{f}(t,t,t,t) = f(t)$ and the d-polar form of given $f(t)$ is unique.

The fact that the given expression works follows from the fact that all the $\hat{f}(\cdot,\cdot,\cdot)$'s are multiaffine, symmetric, and $\hat{f}(t,t,t) = f(t)$.

de Casteljau: Polar Values

Given polar values

$$\hat{f}(r,r,r), \hat{f}(r,r,s), \hat{f}(r,s,s), \hat{f}(s,s,s),$$

find $\hat{f}(t,t,t)$. Clues:

- Use symmetry.
- Polar form is multiaffine, i.e.

  $$\hat{f}(r,r,t) = at + b$$

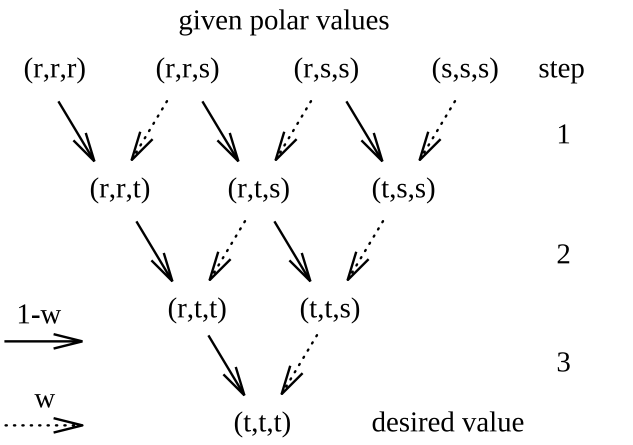
  for some a, b. So, letting $w = (t-r)/(s-r)$,

  $$\begin{aligned}\hat{f}(r,r,t) &= \hat{f}[r,r,r+w(s-r)] \\ &= a[r+w(s-r)] + b[w+(1-w)] \\ &= a[ws+(1-w)r] + b[w+(1-w)] \\ &= w[as+b] + (1-w)[ar+b] \\ &= w\hat{f}(r,r,s) + (1-w)\hat{f}(r,r,r).\end{aligned}$$

Another way to think about the last formula is by thinking of w as the barycentric coordinate of t with respect to r and s. When $t = r$, $w = 0$ and when $t = s$, $w = 1$. When we vary only one polar argument t, the polar form varies linearly ("affinely", to be precise) with t. So, as t "moves" with constant speed from r to s, w "moves" steadily from 0 to 1 and the polar form also "moves" steadily from $\hat{f}(r,r,r)$ to $\hat{f}(r,r,s)$.

We have tacitly assumed that $r \neq s$, otherwise not only would w be ill-defined, but the given four polar values would all have the same polar arguments!

de Casteljau: Polar Values



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We only show the polar arguments in this diagram.

In all the interpolations, we first (implicitly) use symmetry to reorder the arguments of the interpolated polar forms so that the first and second are identical, while the third varies. This third polar argument is r for one input, s for the other. Then, using the interpolation formula of the previous slide, we find the polar value whose third argument is t.

Our strategy is to find $\hat{f}(t,t,t)$, by introducing successively t's as polar arguments. At every step, we are introducing one more polar argument t: the first row has none, the second row has one, etc.

Doesn't that look like the de Casteljau algorithm for finding a point on a cubic curve given the Bézier points of the curve?

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de Casteljau: Curves

To represent cubic curve $\mathbf{p}(t)$ in polar form, find 3-polar form of each coordinate $(x(t), y(t))$ separately:

$$\hat{\mathbf{p}}(t_1, t_2, t_3) = (\hat{x}(t_1, t_2, t_3), \hat{y}(t_1, t_2, t_3)).$$

To specify shape of a cubic $\mathbf{p}(t)$ in terms of polar values, we can be given

$$\hat{\mathbf{p}}(0,0,0), \hat{\mathbf{p}}(0,0,1), \hat{\mathbf{p}}(0,1,1), \hat{\mathbf{p}}(1,1,1),$$

and find $\mathbf{p}(t) = \hat{\mathbf{p}}(t,t,t)$ as shown. But this is de Casteljau algorithm for Bézier curves!

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de Casteljau: Curves

A cubic curve $\mathbf{p}(t)$

- with Bézier points $\mathbf{b}_1, \ldots \mathbf{b}_4$,
- passing through $\mathbf{b}_1$ at time $t = r$, and
- passing through $\mathbf{b}_4$ at time $t = s$,

can be specified in terms of its polar values

$$\begin{aligned} \hat{\mathbf{p}}(r,r,r) &= \mathbf{b}_1 = \mathbf{p}(r) \\ \hat{\mathbf{p}}(r,r,s) &= \mathbf{b}_2 \\ \hat{\mathbf{p}}(r,s,s) &= \mathbf{b}_3 \\ \hat{\mathbf{p}}(s,s,s) &= \mathbf{b}_4 = \mathbf{p}(s). \end{aligned}$$

The points on the curve correspond to diagonal values of the polar form $\hat{\mathbf{p}}(t,t,t)$ which can be found using repeated linear interpolations.

The shape of the curve will be the same, no matter what r and s are (provided, of course, $r \neq s$): this is what we called "alternate parameterizations" during the last lecture. Also, since the curve shape does not depend of r and s, we still call the $\mathbf{b}_i$'s "Bézier" points of the curve (even through r and s might not be 0 and 1, respectively). However, it is important to keep in mind that the Bernstein form of a Bézier curve, i.e.

$$\mathbf{p}(t) = \mathbf{b}_1(1-t)^3 + \mathbf{b}_2 3(1-t)^2 t + \mathbf{b}_3 3(1-t)t^2 + \mathbf{b}_4 t^3$$

only applies to curves which start at $\mathbf{b}_1$ at $t = 0$ and end at $\mathbf{b}_4$ at $t = 1$. Stated differently, the Bernstein polynomials assume a particular parameterization.

Our previous remark on degree raising also shows how, given the four Bézier points of a cubic curve, we can find the five Bézier points of the (degenerate) quartic with the same shape and parameterization as the cubic.

de Casteljau: Subdivision

de Casteljau algorithm in polar form finds $\mathbf{p}(t) = \hat{\mathbf{p}}(t,t,t)$ given

$$\hat{\mathbf{p}}(r,r,r), \hat{\mathbf{p}}(r,r,s), \hat{\mathbf{p}}(r,s,s), \hat{\mathbf{p}}(s,s,s).$$

Along the way, we find $\hat{\mathbf{p}}(\cdot,\cdot,\cdot)$ with polar arguments all combinations of r, s, and t:

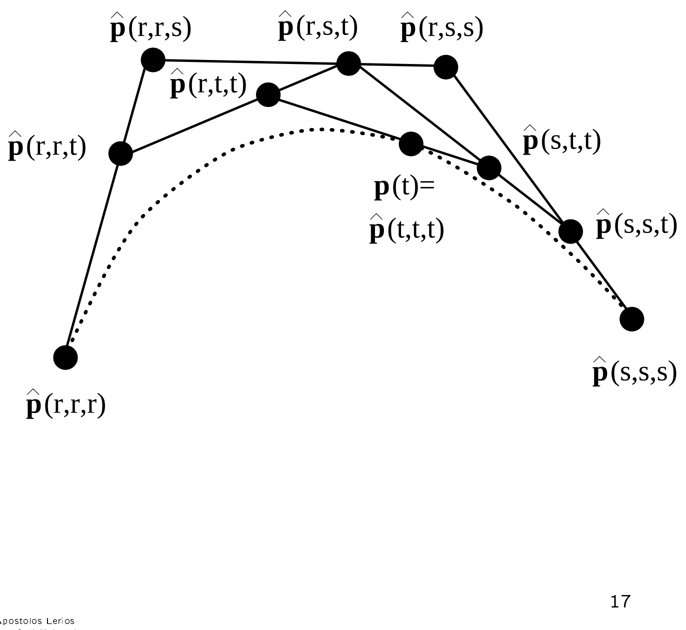
$$\hat{\mathbf{p}}(r,r,r), \hat{\mathbf{p}}(r,r,t), \hat{\mathbf{p}}(r,t,t), \hat{\mathbf{p}}(t,t,t),$$

and

$$\hat{\mathbf{p}}(t,t,t), \hat{\mathbf{p}}(t,t,s), \hat{\mathbf{p}}(t,s,s), \hat{\mathbf{p}}(s,s,s).$$

These are the Bézier points of the left and right subsegments of curve $\mathbf{p}(t)$!

de Casteljau: Subdivision

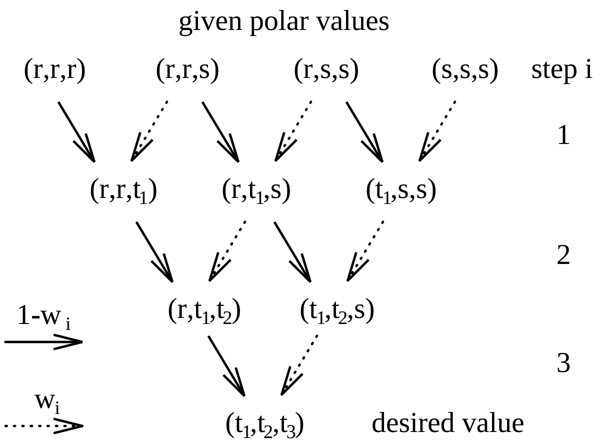


The subdivision property of the de Casteljau algorithm is very important. In the last lecture, we mentioned the convex hull property of the control polygon: if an object (such as a ray of a ray tracer) does not intersect the convex hull of the control polygon, it cannot intersect the curve itself (for $t \in [r, s]$). What do we do if the object *does* intersect the convex hull? We just subdivide the curve into two halves; then, we see which of the two control sub-polygons it intersects! This divide-and-conquer method can be repeated until the control polygons are small enough that they are, effectively, the curve itself. At that point, intersection with the curve can be fully decided by testing intersection against the small control polygons.

Similarly, for rendering Bézier curves, we can render a segment by rendering each of the two sub-segments; we stop when the segment size is a pixel or two, in which case we render the control polygon itself using the graphics pipeline. This method is resolution-sensitive, in contrast to, say, picking a time step dt and approximating the curve with a polyline whose vertices are a distance dt apart. Note that this last method has another important drawback: dt is the distance between the parameter values of adjacent vertices; this is usually a bad approximation of the actual distance between adjacent vertices.

de Boor: Non-Diagonal

Finding $\widehat{f}(t_1, t_2, t_3)$: let $w_i = \frac{t_i - r}{s - r}$.



We are reverting to simple polynomials (or 1D curves, if you prefer) for a while, to make life easier.

The first level of interpolation, with weight w_1, introduces the polar argument t_1, and so on. As the t_i's are at different distances from r and s, the interpolation weights w_i depend on the particular polar argument t_i that we are introducing during step i.

For example, the top left interpolation produces $\widehat{f}(r, r, t_1)$ from $\widehat{f}(r, r, r)$ and $\widehat{f}(r, r, s)$. To do this, we first find

$$w = w_1 = \frac{t_1 - r}{s - r},$$

and then

$$\widehat{f}(r, r, t_1) = w\widehat{f}(r, r, s) + (1 - w)\widehat{f}(r, r, r).$$

There is another possible generalization. We do not need the particular four points we have been using; any four independent points will do...

de Boor: Arbitrary Input

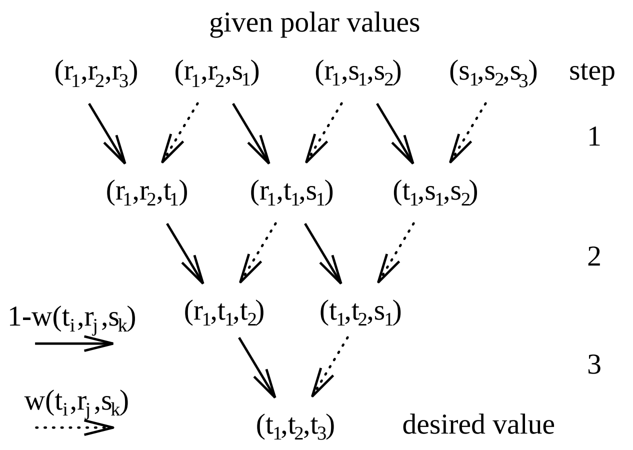
Given polar values

$\widehat{f}(r_1, r_2, r_3), \widehat{f}(r_1, r_2, s_1), \widehat{f}(r_1, s_1, s_2), \widehat{f}(s_1, s_2, s_3)$,

find $\widehat{f}(t_1, t_2, t_3)$. Same technique as before, but

- each interpolation, where t_i interpolates r_j and s_k, uses its own weight
$$w(t_i, r_j, s_k) = \frac{t_i - r_j}{s_k - r_j}.$$
- r_j's and s_k's cannot be arbitrary for interpolation to make sense (i.e. denominator of $w(t_i, r_j, s_k)$ non-zero). Need:
$$\begin{array}{lll} s_1 \neq r_3 & s_2 \neq r_2 & s_3 \neq r_1 \\ s_1 \neq r_2 & s_2 \neq r_1 & \\ s_1 \neq r_1 & & \end{array}$$

de Boor: Arbitrary Input



For example, the top left interpolation produces $\widehat{f}(r_1,r_2,t_1)$ from $\widehat{f}(r_1,r_2,r_3)$ and $\widehat{f}(r_1,r_2,s_1)$. To do this, we first find

$$w = w(t_1,r_3,s_1) = \frac{t_1 - r_3}{s_1 - r_3}$$

which necessitates $s_1 \neq r_3$. Then,

$$\widehat{f}(r_1,r_2,t_1) = w\widehat{f}(r_1,r_2,s_1) + (1-w)\widehat{f}(r_1,r_2,r_3).$$

In total, we do six interpolations, each with its own weight. Each of them imposes a new constraint on the r_j's and s_k's, for a total of six constraints shown on the slide.

de Boor: Curves

When r_j's and s_k's satisfy

$$r_3 \leq r_2 \leq r_1 < s_1 \leq s_2 \leq s_3,$$

then

$\widehat{\mathbf{p}}(r_1,r_2,r_3), \widehat{\mathbf{p}}(r_1,r_2,s_1), \widehat{\mathbf{p}}(r_1,s_1,s_2), \widehat{\mathbf{p}}(s_1,s_2,s_3)$ are the *de Boor* points of the cubic curve $\mathbf{p}(t)$.

The points on the curve correspond to diagonal values of the polar form. They can be found using repeated linear interpolations. This is the *de Boor algorithm*.

Convex hull property: for $t \in [r_1, s_1]$, $\mathbf{p}(t)$ lies in the convex hull of de Boor points.

The condition shown guarantees that all the inequalities of the previous slide are satisfied (actually, it's stronger than that). Put simply, if all the r_j's are smaller than all the s_k's, no r_j can be equal to any s_k.

This construction arises in the context of *B-splines*, i.e. piecewise polynomial curves, where the sequence of r_j's and s_k's is part of the spline's *knot sequence*. We'll talk about B-splines in detail next time.

There is an intuitive reason that the convex hull property holds: when calculating $\mathbf{p}(t)$ with $t \in [r_1, s_1]$ from its de Boor points, all interpolations we perform use weights $w(t,r_j,s_k)$ between 0 and 1 (since t is at least as big as any r_j and no greater than any s_k). Hence, all intermediate points lie between given ones, and $\mathbf{p}(t)$ lies in the convex hull of the de Boor points.

Homogenization: Implicit Curves

A Euclidean curve is a set of Euclidean points $\mathbf{p} = (p_x, p_y)$ which satisfy some equation, e.g. line

$$p_x + 5p_y - 9 = 0.$$

Homogenizing the equation of a Euclidean curve means finding an equation which is satisfied by these points on the projective plane (and no other) whose corresponding Euclidean points, if any, are on the given curve:

$$p_x/p_w + 5p_y/p_w - 9 = 0.$$

And for a well-defined expression for points-at-infinity:

$$p_x + 5p_y - 9p_w = 0.$$

The reason we will go through the mess of homogenizing curves is that homogenization makes life easier when differentiating polar forms (next topic) and when analyzing rational curves (a future topic).

Remember that the Euclidean plane is the regular plane. By adding the points-at-infinity, we get the projective plane. A point on the projective plane is represented by triples $(\lambda p_w; \lambda p_x, \lambda p_y)$, where $\lambda \neq 0$; if $p_w = 0$, it is a point-at-infinity, otherwise it corresponds to the Euclidean point $(p_x/p_w, p_y/p_w)$.

The wording in the definition of homogenization may be ambiguous. The requirement for the homogeneous form of a curve's equation is that no Euclidean points are added or removed from the curve. However, points-at-infinity may be added to the curve at will. Usually, they are added in such a way that the curve is continuous on the projective plane.

Implicit curves are defined in terms of the zero crossings of an equation.

Homogenization: Parametric Curves

Polynomial maps parameter, $t \in \Re$, to $f(t) \in \Re$, e.g.

$$f(t) = 4t^3 + 3t + 5.$$

Can homogenize...

range: $f(t) = (1; 4t^3 + 3t + 5)$.

domain:

$$f[(t_w; t_x)]t_w^3 = 4t_x^3 + 3t_x t_w^2 + 5t_w^3.$$

both:

$$\begin{aligned} f[(t_w; t_x)] &= (1; 4\left(\frac{t_x}{t_w}\right)^3 + 3\frac{t_x}{t_w} + 5) \\ &= (t_w^3; 4t_x^3 + 3t_x t_w^2 + 5t_w^3). \end{aligned}$$

Last step allowed points-at-infinity to map to points-at-infinity.

A polynomial is a parametric curve in 1D. We'll consider curves in higher dimensions soon.

Homogenizing coordinates on the real line is similar to homogenizing the plane. That is, we represent a point $\mathbf{p}$ on the line not just by a single number p but using a pair of real numbers $(p_w; p_x)$. As in the case of the plane, a point on the *projective* line is represented by pairs $(\lambda p_w; \lambda p_x)$, where $\lambda \neq 0$; if $p_w = 0$, it is the point-at-infinity (there is just one), otherwise it corresponds to the point p_x/p_w on the *Euclidean* line.

Homogenizing the domain alone yields an implicit equation which is not solvable if $(t_w; t_x)$ is the point-at-infinity. By homogenizing the range too, the point-at-infinity in the input maps to the point-at-infinity in the output. From now on, we will always be homogenizing both range and domain.

Homogenization: Polar Forms

Polar form maps parameters, $t, r, s \in \Re$, to $\hat{f}(t, r, s) \in \Re$, e.g. polar form of $f(t)$ is

$$\hat{f}(t, r, s) = 4(trs) + (t + r + s) + 5.$$

Homogenizing both range and domain...

$$\begin{aligned} & \hat{f}[(t_w; t_x), (r_w; r_x), (s_w; s_x)] \\ &= (1; 4\frac{t_x}{t_w}\frac{r_x}{r_w}\frac{s_x}{s_w} + \frac{t_x}{t_w} + \frac{r_x}{r_w} + \frac{s_x}{s_w} + 5) \\ &= (\hat{f}_w; \hat{f}_x) \end{aligned}$$

where

$$\begin{aligned} \hat{f}_w &= t_w r_w s_w \\ \hat{f}_x &= 4 t_x r_x s_x + t_x r_w s_w + r_x t_w s_w + s_x t_w r_w \\ & \quad + 5 t_w r_w s_w. \end{aligned}$$

This is polar form of *homogenized* $f(t)$!

The last remark means that polarization and homogenization can take place in either order: the result will be the same whether we homogenize the polar form (as we did in this slide) or polarize the homogeneous form (not shown, and it's not as as easy as polarizing the non-homogenous form).

Homogenization: Polar Forms

Polar form of planar curve maps $t, r, s \in \Re$, to $\hat{\mathbf{p}}(t, r, s) \in \Re^2$, e.g.

$$\hat{\mathbf{p}}(t, r, s) = (4trs, t + r + s).$$

Homogenizing both range and domain...

$$\begin{aligned} & \hat{\mathbf{p}}[(t_w; t_x), (r_w; r_x), (s_w; s_x)] \\ &= (1; 4\frac{t_x}{t_w}\frac{r_x}{r_w}\frac{s_x}{s_w}, \frac{t_x}{t_w} + \frac{r_x}{r_w} + \frac{s_x}{s_w}) \\ &= (\hat{p}_w; \hat{p}_x, \hat{p}_y) \end{aligned}$$

where

$$\begin{aligned} \hat{p}_w &= t_w r_w s_w \\ \hat{p}_x &= 4 t_x r_x s_x \\ \hat{p}_y &= t_x r_w s_w + r_x t_w s_w + s_x t_w r_w. \end{aligned}$$

Every term contains *either* t_w *or* t_x. Similarly for $(r_w; r_x)$ and $(s_w; s_x)$.

This last observation follows trivially from the way we got rid of the denominators. We multiplied out by the product of the weights. Thus, terms containing the ratio t_x/t_w lost their denominator t_w but kept t_x, while all the other terms got a t_w. Make a note of this remark; we will use it later on. Also, remark that none of t_w, t_x, etc. appears raised to a power other than unity or zero; this is due to the multiaffine character of the polar form.

Differentiation: Claim

Let $\delta = (0; 1)$.

We will show that the ith derivative $\mathbf{p}^{(i)}(t)$ of a cubic curve is given by a scalar multiple of its polar form $\widehat{\mathbf{p}}(\cdot,\cdot,\cdot)$ evaluated with i polar arguments equal to δ and $3 - i$ equal to t.

Gameplan:

- Will show two lemmata.
- Will prove above theorem.

As usual, our result is trivially extensible to curves of arbitrary degree.

Differentiation: Lemma 1

$$\widehat{f}[(1; t+h), \cdot, \cdot] = \widehat{f}[(1; t), \cdot, \cdot] + \widehat{f}[(0; h), \cdot, \cdot].$$

Why? Expand LHS into terms where

- $t_w = 1$ appears: show up in $\widehat{f}[(1; t), \cdot, \cdot]$.
- $t_x = t$ appears: in $\widehat{f}[(1; t), \cdot, \cdot]$.
- $t_x = h$ appears: in $\widehat{f}[(0; h), \cdot, \cdot]$. $\widehat{f}[(0; h), \cdot, \cdot]$ contains no others since it sets $t_w = 0$.

Example:

$$\begin{aligned}
&\widehat{f}[(t_w; t_x), (r_w; r_x), (s_w; s_x)] \\
&\quad = (t_w r_w s_w; t_x r_w s_w + r_x t_w s_w + s_x t_w r_w), \\
&\widehat{f}[(1; t+h), \cdot, \cdot] \\
&\quad = (r_w s_w; t r_w s_w + h r_w s_w + r_x s_w + s_x r_w), \\
&\widehat{f}[(1; t), \cdot, \cdot] = (r_w s_w; t r_w s_w + r_x s_w + s_x r_w), \\
&\widehat{f}[(0; h), \cdot, \cdot] = (0; h r_w s_w).
\end{aligned}$$

The dots stand for "anything", in the following sense: whatever was there in the previous equation, stays there. In other words, they save us from carrying around $(r_w; r_x)$ and $(s_w; s_x)$.

LHS stands for "Left-Hand Side", i.e. $\widehat{f}[(1; t+h), \cdot, \cdot]$.

We have assumed that $h \neq 0$, otherwise $(0; h)$ would not be a valid point on the projective line. More importantly, in order to add $\widehat{f}[(1; t), \cdot, \cdot]$ and $\widehat{f}[(0; h), \cdot, \cdot]$, we now have to abolish our understanding of $(0; h)$ as the point-at-infinity on the projective line: while $\widehat{f}[(0; h), \cdot, \cdot]$ yields the same point on the projective line no matter what $h \neq 0$ is, the sum $\widehat{f}[(1; t), \cdot, \cdot] + \widehat{f}[(0; h), \cdot, \cdot]$, defined by adding pairwise the components $\widehat{f}_w$ and $\widehat{f}_x$ of $(\widehat{f}_w; \widehat{f}_x)$, produces different results depending on h. The proper interpretation from now on for both arguments and results is that of *sites* whose weight can only be 0 or 1.

Recall the observations we made earlier: every term of $\widehat{f}[(t_w; t_x), (r_w; r_x), (s_w; s_x)]$ contains either t_w or t_x, and similarly for $(r_w; r_x)$ and $(s_w; s_x)$; hence the list on the slide accounts for all terms without repetitions.

Also, remember that none of t_w, t_x, etc. appears raised to a power other than unity or zero; hence, each term of the LHS containing $t_x = t + h$ can be split into two terms.

Differentiation: Lemma 2

$$\hat{f}[(0;h),\cdot,\cdot] = h\hat{f}(\delta,\cdot,\cdot).$$

Why? Conceptually divide the LHS terms into two groups:

- $t_w = 0$ appears: since $t_w = 0$, these all go away.
- $t_x = h$ appears: factor h out leaving $\hat{f}(\delta,\cdot,\cdot)$.

Example:

$$\begin{aligned} &\hat{f}[(t_w;t_x),(r_w;r_x),(s_w;s_x)] \\ &\quad = (t_w r_w s_w; t_x r_w s_w + r_x t_w s_w + s_x t_w r_w), \\ &\hat{f}[(0;h),\cdot,\cdot] = (0; h r_w s_w), \\ &h\hat{f}(\delta,\cdot,\cdot) = h\hat{f}[(0;1),\cdot,\cdot] = h(0; r_w s_w). \end{aligned}$$

Differentiation: Theorem

$$\begin{aligned} \mathbf{p}'(t) &= \lim_{h\to 0} \frac{\mathbf{p}(t+h)-\mathbf{p}(t)}{h} \\ &= \lim_{h\to 0} \frac{\hat{\mathbf{p}}(t+h,t+h,t+h)-\hat{\mathbf{p}}(t,t,t)}{h}. \end{aligned}$$

Lemmata combined show

$$\hat{\mathbf{p}}(t+h,\cdot,\cdot) = \hat{\mathbf{p}}(t,\cdot,\cdot) + h\hat{\mathbf{p}}(\delta,\cdot,\cdot).$$

By repeated application,

$$\begin{aligned} \hat{\mathbf{p}}(t+h,t+h,t+h) &= \hat{\mathbf{p}}(t,t,t) + 3h\hat{\mathbf{p}}(t,t,\delta) \\ &+ 3h^2\hat{\mathbf{p}}(t,\delta,\delta) + h^3\hat{\mathbf{p}}(\delta,\delta,\delta) \end{aligned}$$

Substituting above

$$\begin{aligned} \mathbf{p}'(t) &= \lim_{h\to 0}[3\hat{\mathbf{p}}(t,t,\delta) + O(h) + O(h^2)] \\ &= 3\hat{\mathbf{p}}(t,t,\delta). \end{aligned}$$

We have quietly extended the previous results to curves in higher dimensions than 1D. The mechanics of this extension are trivial.

We freely write $\hat{\mathbf{p}}(\cdot,\cdot,\cdot)$ without specifying whether we are using the homogenized or the non-homogeneous polar form. There are three reasons for allowing this sloppiness:

- We can always tell from context which form is needed. Context includes the type of polar arguments, as well as surrounding terms like other summands.
- It is trivial to convert the polar argument t to the homogenous $(1;t)$ when needed.
- And, of course, both forms yield the same value if the polar arguments are all Euclidean points.

For example, the second formula should have been written as

$$\hat{\mathbf{p}}[(1;t+h),\cdot,\cdot] = \hat{\mathbf{p}}[(1;t),\cdot,\cdot] + h\hat{\mathbf{p}}(\delta,\cdot,\cdot).$$

The $O(h)$ notation means "order of h", i.e. some term which grows as fast as h, and thus, as $h \to 0$, it vanishes. Similarly for $O(h^2)$.

Differentiation: General

For a curve $\mathbf{p}(t)$ of degree d,

$$\mathbf{p}'(t) = d\hat{\mathbf{p}}(\underbrace{t,\ldots,t}_{d-1},\delta),$$

and by repeated differentiation,

$$\mathbf{p}^{(i)}(t) = d(d-1)\cdots(d-i+1)\hat{\mathbf{p}}(\underbrace{t,\ldots,t}_{d-i},\underbrace{\delta,\ldots,\delta}_{i}).$$

Also, d steps of differentiation yield constant

$$\mathbf{p}^{(i)}(t) = d!\hat{\mathbf{p}}(\underbrace{\delta,\ldots,\delta}_{d}).$$

Finally, from lemmata, and for cubics,

$$\mathbf{p}'(t) = \frac{3}{h}[\hat{\mathbf{p}}(t,t,t+h) - \hat{\mathbf{p}}(t,t,t)].$$

The motivation behind the last rewriting is to express the derivative in terms of non-homogeneous polar forms.

Differentiation: Endpoints

Last lecture showed

$$\begin{aligned} \mathbf{p}'(0) &= 3(\mathbf{b}_2 - \mathbf{b}_1) \\ \mathbf{p}'(1) &= 3(\mathbf{b}_4 - \mathbf{b}_3) \end{aligned}$$

where the $\mathbf{b}_i$'s are the Bézier points of the curve $\mathbf{p}(t)$, with $\mathbf{p}(0) = \mathbf{b}_1$ and $\mathbf{p}(1) = \mathbf{b}_4$.

Polar form yields same:

$$\begin{aligned} \mathbf{p}'(0) &= 3[\widehat{\mathbf{p}}(0,0,1) - \widehat{\mathbf{p}}(0,0,0)] \\ \mathbf{p}'(1) &= 3[\widehat{\mathbf{p}}(1,1,1) - \widehat{\mathbf{p}}(1,1,0)], \end{aligned}$$

and generalizes to

$$\begin{aligned} \mathbf{p}'(r) &= \frac{3}{s-r}[\widehat{\mathbf{p}}(r,r,s) - \widehat{\mathbf{p}}(r,r,r)] \\ \mathbf{p}'(s) &= \frac{3}{s-r}[\widehat{\mathbf{p}}(s,s,s) - \widehat{\mathbf{p}}(s,s,r)]. \end{aligned}$$

- For the derivative at $t = 0$, we used $h = 1$ in the last formula of the previous slide.
- For the derivative at $t = 1$, we used $h = -1$.
- For the derivative at $t = r$, we used $h = s - r$.
- For the derivative at $t = s$, we used $h = r - s$.

Why Polar Forms?

- Aren't Bernstein polynomials sufficient? They are. Most literature on splines is in this form.
- Polar forms make some proofs much easier.
- They also make some concepts more intuitive: they are best for people who "think pictures", not "formulas".
- They make the transition from individual curves to splines very gentle.

Wait till next time for splines...

References

- Guibas/Ramshaw (handout 15), sections 2, 3, 4.1, 5.
- Guibas/Ramshaw (handout 19), section 1 (up to 1.4).
- Ramshaw: sections 1–6, 9–11.
- Seidel: pages 38–42.
- Farin: pages 36–39, 44, 56–58, 77–78, 181–183.

They are all in the course reader, except for Farin, which is on reserve at the MCS library.

The Guibas and Ramshaw handouts are from the 1992 lecture notes of CS 348A; they are well written, and they constitute a very gentle introduction to polar forms without assuming any prior background on Bernstein polynomials. Ramshaw's work is (possibly overwhelmingly) theoretical but precise; he also proposes some standards for notation and nomenclature (which we did not follow). Seidel is utterly concise, and shows the transition from the Bernstein polynomials to the polar forms; it's a great summary of the essential concepts, but unreadable for first-timers. Farin follows the same approach as Seidel: from the Bernstein basis to the polar forms.