## Announcements

- Course reader: Monday 10/9.
- Topic assignments may change.
- Field trip: to PDI on Thursday 10/19.
- Thursday: light bursts. *CIS*.
- Tuesday: representing polynomials. *CIS*.





There will be one course reader on reserve at the MCS library. Extra copies of the reader were ordered for auditors. Thursday's papers are being handed out today.

Topic assignments may change if people drop the class and some topics are left assigned to one person while others to three. The fact that the university has moved the drop deadline means the matter will be settled soon.

The field trip will be during class-time. Students who own buses, please let me know!

Is Alice Tull here?

All color figures from one of the papers are on the WWW. I also brought the papers to class to see the pictures.





## Today

- Site notation.
- Barycentric coordinates.
- Homogeneous coordinates.
- Projective geometry.





## Site Notation

Our world is the *plane*.

Usually:

- Point $\mathbf{p}$ represented as $(p_x, p_y)$.
- Vector $\vec{v}$ represented as $(v_x, v_y)$.

*Site notation*:

- $\mathbf{p}$ represented as $(1; p_x, p_y)$.
- $\vec{v}$ represented as $(0; v_x, v_y)$.

Initial "1" or "0" is *weight*.

A *site* is either a point or a vector.

Our world is the plane just like Edwin Abbott's *Flatland*; read it! Extending our discussion to 3D space is very straightforward.

The $p_x$, $p_y$, $v_x$, $v_y$ are real numbers.

For now, weight will 1 or 0, nothing else. I am dictatorially defining my notation. You'll see the rationale behind this definition later on.





## Sites: Rationale

It's convenient:

- Keeps points, vectors distinct.
- Simplifies formulas.

Examples of use:

- $\mathbf{p}_1 - \mathbf{p}_2 = (0; \cdot, \cdot)$, a vector!
- $\mathbf{p}_1 + \mathbf{p}_2 = (2; \cdot, \cdot)$, an illegal operation!
- $\mathbf{p} + \vec{v} = (1; \cdot, \cdot)$, a point!
- ... and more.





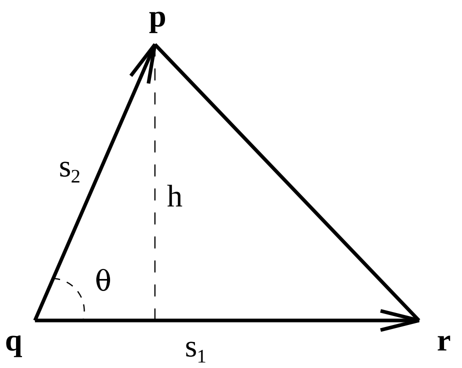
Reminder for your presentations: I can have formulas in my lectures, you cannot.

The first three examples illustrate site notation's usefulness for distinguishing points and vectors. The next example illustrates how site notation simplifies formulas...





## Sites: Triangle Area




Area $A = 0.5|\vec{s}_1|h =$

$$= 0.5|\vec{s}_1||\vec{s}_2| \sin\theta = 0.5|\vec{s}_1 \times \vec{s}_2|$$

$$= 0.5 \text{ abs} \begin{vmatrix} r_x - q_x & r_y - q_y \\ p_x - q_x & p_y - q_y \end{vmatrix} = 0.5 \text{ abs} \begin{vmatrix} 1 & q_x & q_y \\ 1 & r_x & r_y \\ 1 & p_x & p_y \end{vmatrix}$$

$\times$ stands for the cross product operation. In order to apply it to 2D vectors, assume the triangle $\mathbf{pqr}$ lies on the $xy$ plane, and extend $\vec{s}_1$ and $\vec{s}_2$ to 3D (with a 0 $z$-component).

Missing steps in derivation:

$$\begin{aligned}
\vec{s}_1 &= (r_x - q_x, r_y - q_y, 0) \\
\vec{s}_2 &= (p_x - q_x, p_y - q_y, 0) \\
\vec{s}_1 \times \vec{s}_2 &= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ r_x - q_x & r_y - q_y & 0 \\ p_x - q_x & p_y - q_y & 0 \end{vmatrix} \\
&= \begin{vmatrix} 1 & q_x & q_y \\ 0 & r_x - q_x & r_y - q_y \\ 0 & p_x - q_x & p_y - q_y \end{vmatrix},
\end{aligned}$$

where $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ are the unit vectors along the coordinate axes.





## Sites: Signed Area

**Unsigned area:**

$$A = 0.5 \text{ abs} \begin{vmatrix} 1 & q_x & q_y \\ 1 & r_x & r_y \\ 1 & p_x & p_y \end{vmatrix}.$$

**Signed area:**

$$A_s = 0.5 \begin{vmatrix} 1 & q_x & q_y \\ 1 & r_x & r_y \\ 1 & p_x & p_y \end{vmatrix}.$$

$A_s < 0$**:** $\mathbf{q}$, $\mathbf{r}$, $\mathbf{p}$ in clockwise order.

$A_s = 0$**:** $\mathbf{q}$, $\mathbf{r}$, $\mathbf{p}$ collinear.

$A_s > 0$**:** $\mathbf{q}$, $\mathbf{r}$, $\mathbf{p}$ in counterclockwise order.





The sign is trivially evident if we think of the cross product formulation: the order of the vertices determines the sign of the $z$ component of the cross product.





## Sites: Lines

If $A_s = 0$, $\mathbf{q}$, $\mathbf{r}$, $\mathbf{p}$ collinear. So $\mathbf{q} = (1; x, y)$ is on line through $\mathbf{r}$ and $\mathbf{p}$ iff

$$\begin{vmatrix} 1 & x & y \\ 1 & r_x & r_y \\ 1 & p_x & p_y \end{vmatrix} = 0.$$

Also, $\mathbf{q} = (1; x, y)$ is on line through $\mathbf{r}$ and in the direction of $\vec{v} = (0; v_x, v_y)$ iff

$$\begin{vmatrix} 1 & x & y \\ 1 & r_x & r_y \\ 0 & v_x & v_y \end{vmatrix} = 0$$

since this condition is the same as

$$x v_y - y v_x - \begin{vmatrix} r_x & r_y \\ v_x & v_y \end{vmatrix} = 0.$$

"Iff" means "if and only if".

There is a pattern here: write out relevant sites one below the other!

For example, take $\mathbf{r}$ to be the origin. In this case, the equation is trivially satisfied. Since $\mathbf{r}$ affects only the third ("constant") term, it changes the $y$-intercept of the line, not its direction.

Summary: we introduced the site notation, and saw examples:

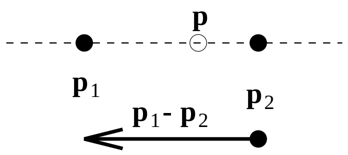
- where notation prevents mistakes due to confusion between points and vectors, and
- where it simplifies formulas.

We will now use the site notation to define a new form of "coordinates", i.e. a new way to denote the position of points on the plane.





## Sites: Barycentric Combinations




$\mathbf{p} = a\mathbf{p}_1 + b\mathbf{p}_2 = (a+b; \cdot, \cdot)$ is point iff $a+b=1$. So, simplify to $\mathbf{p} = a\mathbf{p}_1 + (1-a)\mathbf{p}_2 = \mathbf{p}_2 + a(\mathbf{p}_1 - \mathbf{p}_2)$. Note:

- $a$ scales vector $\mathbf{p}_1 - \mathbf{p}_2$.
- Varying $a$ gives points $\mathbf{p}$ on line through $\mathbf{p}_1$ and $\mathbf{p}_2$.

Inversely, given $\mathbf{p}$, $\mathbf{p}_1$ and $\mathbf{p}_2$ and finding $a$ is expressing $\mathbf{p}$ as *barycentric combination* of $\mathbf{p}_1$ and $\mathbf{p}_2$. New way to express coordinates of points along a given line: $a$ alone!





Assume $\mathbf{p}_1$ and $\mathbf{p}_2$ distinct.

- $a = 0$ yields $\mathbf{p}_2$,
- $a = 1$ yields $\mathbf{p}_1$,
- $a = 0.5$ is midpoint (averaging $\mathbf{p}_1$ and $\mathbf{p}_2$),
- $a < 0$ or $a > 1$ yield points outside segment.

Barycentric combinations are also called *affine* or *linear* combinations.

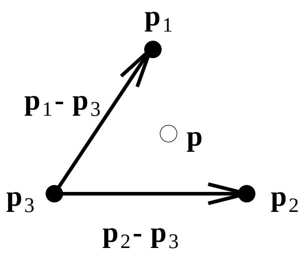
We have one unknown ($a$) and two equations (one each for $x$, $y$ components), so the system of equations is overdetermined: it has no solution iff $\mathbf{p}$ does not lie along given line.





## Sites: Barycentric Coordinates

In general, $\mathbf{p} = a_1\mathbf{p}_1 + \cdots + a_n\mathbf{p}_n$ is a point iff $a_1 + \cdots + a_n = 1$. Consider $n = 3$.




If $\mathbf{p} = a\mathbf{p}_1 + b\mathbf{p}_2 + c\mathbf{p}_3$, where $a+b+c=1$, then $(a,b,c)$ are *barycentric coordinates* of $\mathbf{p}$ w.r.t. respect to $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. Simpler form:

$$\mathbf{p} = \mathbf{p}_3 + a(\mathbf{p}_1 - \mathbf{p}_3) + b(\mathbf{p}_2 - \mathbf{p}_3).$$

The first sum is the general form of a barycentric combination.

"w.r.t." means "with respect to".

Assume $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ are not collinear.

Graphically, $a$ and $b$ scale respective vectors $\mathbf{p}_1 - \mathbf{p}_3$ and $\mathbf{p}_2 - \mathbf{p}_3$.

Inverse problem: given $\mathbf{p}$, $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, find barycentric coordinates of $\mathbf{p}$ with respect to $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. The resulting system has three unknowns ($a$, $b$, $c$) and three equations (one for each of the $x$, $y$ components, and $a + b + c = 1$), so it is always solvable!

Remember site notation's restriction for the weight to be 0 or 1? Now, we will let the weight be anything, and see if we can somehow interpret the resulting triples of numbers.





## Homogeneous Coordinates

Consider all triples $(w; x, y)$ such that

- Not *all* $w$, $x$, $y$ are zero.
- If $w \neq 0$, triple maps to *Euclidean* point $(x/w, y/w)$.
- If $w = 0$, triple maps to *point-at-infinity* or *direction* $(x, y)$.

All Euclidean points: *Euclidean* plane.

$+=$ all points-at-infinity: *projective* plane.





If $w$, $x$, $y$ are all zero, the triple means nothing.

Euclidean points are the regular points on the plane. We'll just call them "points" from now on.

Comparing homogeneous coordinates to the site notation, we see that $w \neq 0$ still represents points, and $w = 0$ represents other entities of the plane: in site notation, $w = 0$ represents vectors, and in homogeneous coordinates it represents some other entities which we call *points-at-infinity*. The dashes in "point-at-infinity" are nonstandard.





## Homogeneous Coordinates: Remarks

- One point or point-at-infinity $\mathbf{p}$ has many triples all of the form $(\lambda w; \lambda x, \lambda y)$ for all $\lambda \neq 0$. These are *homogeneous coordinates* of $\mathbf{p}$.
- If $w = 1$, $(1; x, y)$ are *normalized coordinates*.

Homogenization adds a weight coordinate, while normalization removes it. For those of you who took CS 248, this should sound very familiar.

We will justify the use of homogeneous coordinates later; it won't make any sense if we try to do so right now. So, I will first show their use, and then comment on their advantages over other notations.





## Homogeneous Coordinates: Lines

A line $[a; b, c]$ on projective plane is set of points $(w; x, y)$ such that $aw + bx + cy = 0$. Note:

- All triples of form $[\lambda a; \lambda b, \lambda c]$ for all $\lambda \neq 0$ represent same line.
- *Line-at-infinity* $[a; 0, 0]$ contains all points-at-infinity.
- Every line contains one (or all) point-at-infinity. e.g. $[3; 4, 5]$ contains $(0; -5, 4)$.
- Parallel lines intersect at points-at-infinity. e.g. $[1; 2, 3]$ and $[1; 4, 6]$ meet at $(0; -3, 2)$.





Not all of $a$, $b$, and $c$ can be zero at the same time.

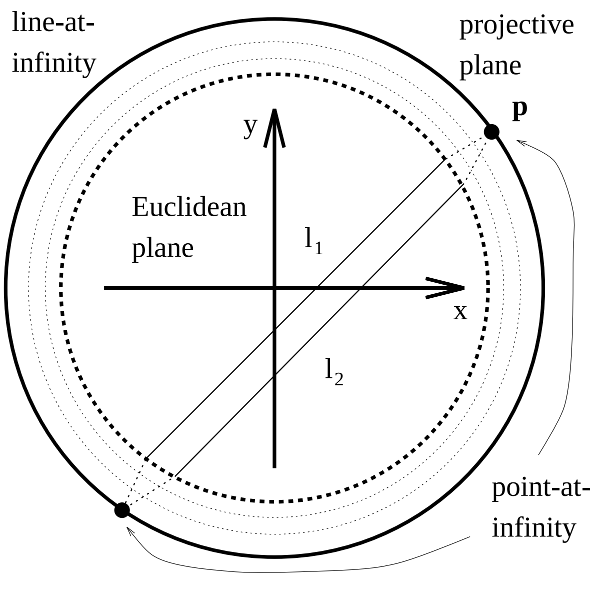
Which point-at-infinity does a given line $[a; b, c]$ contain? Solve $bx + cy = 0$ for known $b$, $c$.

Note that when we say "the line $[3; 4, 5]$" what we really mean is "the line whose one of infinite triples is $[3; 4, 5]$." Similarly for points.





## Homogeneous Coordinates: Visualization



This is one of Stolfi's visualizations of the projective plane: we are showing the *straight* model. Others are just as good visualizations for the same abstraction, useful in different domains. The assignment will introduce the spherical model.

$\mathbf{p}$ is the intersection point-at-infinity of the parallel lines $l_1$ and $l_2$.





## Homogeneous Coordinates: Duality

Can look at equation $aw + bx + cy = 0$ in two ways:

- Given *line* $[a; b, c]$, its *points* $(w; x, y)$ must satisfy equation.
- Given *point* $(w; x, y)$, all *lines* $[a; b, c]$ through that point must satisfy equation.





Everything has two — dual — interpretations: formulas, theorems, the O.J. Simpson trial... An example of a dual interpretation follows...





## Homogeneous Coordinates: Intersecting Lines

Consider equations:

$$a_1 w + b_1 x + c_1 y = 0$$
$$a_2 w + b_2 x + c_2 y = 0$$

Dual interpretations:

- At what point $(w; x, y)$ do lines $[a_1; b_1, c_1]$ and $[a_2; b_2, c_2]$ meet?
- Which line $[w; x, y]$ passes through points $(a_1; b_1, c_1)$ and $(a_2; b_2, c_2)$?

Solving one problem solves the other!

## Homogeneous Coordinates: Maps

*Maps* associate every element of the plane $\mathbf{p}$ with some element $\mathbf{q}$.

**Linear maps:** can find matrix $\mathbf{A}$ such that

$$\mathbf{q} = \mathbf{A}\mathbf{p}.$$

**Affine maps:** can find $\mathbf{A}$ and vector $\vec{t}$ such that

$$\mathbf{q} = \mathbf{A}\mathbf{p} + \vec{t}.$$

*Projective* maps are *linear* maps on the *projective* plane:

$$\begin{pmatrix} p_w \\ p_x \\ p_y \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} q_w \\ q_x \\ q_y \end{pmatrix}.$$





In their most general form, maps associate every entity of some space to entities of some other space. When the source and target space are the same, we say that a map is defined *on* that space. We will now focus on maps defined on the plane, whether Euclidean or projective. If such maps can be written in a concise form using a few parameters (a matrix and, optionally, a vector here), they are given special names.

Projective maps are linear maps on the projective plane. They may map Euclidean points to points-at-infinity, or the other way around. When we ignore points-at-infinity, and we only look at the (partial) correspondence between Euclidean points, we get a (partial) map of the Euclidean plane. Such "projective maps" of the Euclidean plane show up in the graphics pipeline (the viewing transform causing perspective foreshortening). If we express these maps in terms of the Cartesian coordinates of the Euclidean points, the resulting expression does not have the elegance of the linear form; such elegant formulas are important because they are easy to implement, esp. in hardware, and they also provide much easier notation to work with.





## Homogeneous Coordinates: Rationale

- Simpler formulas.
- Fewer special cases.
- Unification and extension of concepts.
- Duality: every theorem works when swapping the words "point" and "line".





Corresponding examples:

- Projective maps.
- All distinct lines intersect.
- Projective maps.
- Two distinct lines define a point (at their intersection), and two distinct points define a line (through them). Also, all the formulas are symmetric.

# References

- Farin: pages 13–21.
- Böhm: sections 2–3.
- Stolfi: chapter 1.





Where are they?

- Library reserve.
- Course reader.
- Course reader. The whole book is on reserve, but the chapters which are not in the reader are very theoretical.