Rotations and Quaternions

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December 13, 1995

1 Representing Rotations with Matrices

1.1 The Theory

A rotation in 2D (on the plane), about the origin by an angle θ, is represented by the matrix

$$\mathbf{R}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

The sign of θ can be determined by examining the effect of this matrix on two points, one along each principal axis. The point $(1;1,0)$ is transformed to the point $(1;\cos\theta,\sin\theta)$, while $(1;0,1)$ is mapped onto $(1;-\sin\theta,\cos\theta)$. Since the rotation matrix has a single parameter, namely θ, plane rotations have a single degree of freedom (d.o.f.).

In 3D, it is easy to derive the rotation matrices about the principal axes x, y, and z. First, a rotation about the z axis moves the points on the xy plane in the same way as the plane rotation matrix $\mathbf{R}_2$. In other words, the 3D points subjected to the rotation retain their z coordinate, while their x and y coordinates are turned about the origin (or the z axis). Hence the rotation about the z axis by an angle χ, is represented by the matrix

$$\mathbf{R}_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\chi & -\sin\chi & 0 \\ 0 & \sin\chi & \cos\chi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The rotation matrices about the other two axes are obtained by switching around the axes. To get $\mathbf{R}_x$, we must realize that the x component of the points is not altered; meanwhile, the yz plane now plays the role that the xy plane played when we rotated about the z axis. Therefore, the rotation matrix about the x axis by an angle ϕ, is

$$\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

Finally, the rotation matrix about the y axis by an angle ψ, is

$$\mathbf{R}_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & 0 & \sin\psi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\psi & 0 & \cos\psi \end{pmatrix}.$$

A general rotation in 3D can be achieved by combining the effect of the rotation matrices $\mathbf{R}_x$, $\mathbf{R}_y$, and $\mathbf{R}_z$ in a single matrix $\mathbf{R}_3 = \mathbf{R}_z\mathbf{R}_y\mathbf{R}_x$. The resulting matrix will contain three parameters, namely χ, ϕ, and ψ. It follows that 3D rotations have three d.o.f.

There are other ways to combine three rotations about the principal axes. For example, we could first rotate about z by χ_1, then about y by ψ, and finally about z again by χ_2. These three angles, χ_1, ψ, and χ_2, are called *Euler angles.* See [CP66] for more information on this representation of rotations. Note also that the term *Euler angles* is often used for any set of three angles that can represent a rotation if applied in some order about the principal axes. Thus, χ, ϕ, and ψ are also called Euler angles by some authors. [KK68] lists 12 other conventions encountered in the literature.

The $\mathbf{R}_3$ matrix possesses several important properties. In order to present them concisely, we will represent $\mathbf{R}_3$ by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u_x & v_x & w_x \\ 0 & u_y & v_y & w_y \\ 0 & u_z & v_z & w_z \end{pmatrix}.$$

The properties of $\mathbf{R}_3$ are[1]:

$$\begin{aligned} u_x^2 + u_y^2 + u_z^2 &= 1 \quad (1) \\ v_x^2 + v_y^2 + v_z^2 &= 1 \quad (2) \\ w_x^2 + w_y^2 + w_z^2 &= 1 \quad (3) \\ u_x v_x + u_y v_y + u_z v_z &= 0 \quad (4) \\ u_x w_x + u_y w_y + u_z w_z &= 0 \quad (5) \\ w_x v_x + w_y v_y + w_z v_z &= 0 \quad (6) \\ \det \mathbf{R}_3 &= 1. \quad (7) \end{aligned}$$

These properties pose a set of six constraints on the nine variable elements of $\mathbf{R}_3$. It follows that $\mathbf{R}_3$ has three d.o.f., as stated above. Moreover, we can conclude that if an arbitrary matrix $\mathbf{R}$ satisfies the above list of properties, then $\mathbf{R}$ is a rotation matrix. Note that while constraint 7 appears to be a seventh constraint on $\mathbf{R}$, it actually isn't. The reason is that constraints 1–6 guarantee that the determinant of $\mathbf{R}$ will be ± 1. If the determinant is -1, $\mathbf{R}$ is a *reflection* matrix, otherwise, it is a *rotation* matrix.

[1] We are listing only the column-based properties of $\mathbf{R}_3$. Similar properties hold for the rows.

Finally, the effect of $\mathbf{R}_3$ on the principal axes allows us to visualize the rotation effected by $\mathbf{R}_3$:

$$\begin{array}{rcl}(1;1,0,0) & \mapsto & (1;u_x,u_y,u_z)\\ (1;0,1,0) & \mapsto & (1;v_x,v_y,v_z)\\ (1;0,0,1) & \mapsto & (1;w_x,w_y,w_z).\end{array}$$

1.2 Problems of the Matrix Representation

An alternative method of representing rotations is by defining the axis of rotation, and the angle of rotation θ. The axis of rotation is a 3D line, but it will be more convenient to use a directed line, or a unit vector $\vec{\alpha}$, instead. We will then adopt the right hand rule for deciding on the sign of θ: aligning our thumb with the rotation axis $\vec{\alpha}$, the digit points in the direction of positive θ.

The 3D matrices we saw above, namely $\mathbf{R}_x$, $\mathbf{R}_y$, and $\mathbf{R}_z$, all represent rotations about one of the principal axes. For example, $\mathbf{R}_x$ represents a rotation about axis $(0;1,0,0)$ and angle ϕ. From the matrix representation of a general rotation, it is easy to derive the new location of the principal axes, as we saw above. However, it is very hard to identify the axis and angle or rotation. An alternative to the matrix representation of rotations, which allows us to identify the rotation axis and angle very easily, is the *quaternion* representation.

2 Representing Rotations with Quaternions

2.1 The Basics of the Theory

A quaternion is a list of 4 numbers a, b, c, d, which we usually write as (a,b,c,d). Alternatively, we can introduce the placeholders I, J, K and represent the same quaternion as $a+bI+cJ+dK$. Some books also use the notation $\dot{q}$ to denote a quaternion. Also, they use q to stand for the so-called "real" part of $\dot{q}$ — what we denoted by a. And they use $\vec{q}$ for the "imaginary" part of $\dot{q}$, i.e. the triplet (b,c,d). Hence, $\dot{q}$ is written as $q+\vec{q}$, and $(a,0,0,0)$ is written simply as the real number a. This multiplicity of notations seems to be quite redundant at first, but it comes in handy when quaternion operations and properties are written down.

There are a lot of operations defined on quaternions. Letting $\dot{q}=(q_0,q_x,q_y,q_z)$, and $\dot{p}=(p_0,p_x,p_y,p_z)$, we define the following:

Scalar multiplication: For any real number s, $s\dot{q}=(sq_0,sq_x,sq_y,sq_z)$.

Norm: $|\dot{q}|=\sqrt{q_0^2+q_x^2+q_y^2+q_z^2}$.

Conjugate: $\overline{\dot{q}}=(q_0,-q_x,-q_y,-q_z)=q-\vec{q}$.

Addition: $\dot{q}+\dot{p}=(q_0+p_0,q_x+p_x,q_y+p_y,q_z+p_z)$.

Dot product: $\dot{q}\cdot\dot{p}=q_0p_0+q_xp_x+q_yp_y+q_zp_z=qp+\vec{q}\cdot\vec{p}$.

Multiplication: $\dot{q}\dot{p} = (a, b, c, d)$ where

$$\begin{aligned}
a &= q_0 p_0 - q_x p_x - q_y p_y - q_z p_z \\
b &= q_0 p_x + q_x p_0 + q_y p_z - q_z p_y \\
c &= q_0 p_y - q_x p_z + q_y p_0 + q_z p_x \\
d &= q_0 p_z + q_x p_y - q_y p_x + q_z p_0 .
\end{aligned}$$

Inverse: $\dot{q}^{-1}$ such that $\dot{q}^{-1}\dot{q} = 1$.

Conjugation or Composite product: $C_{\dot{q}}(\dot{p}) = \dot{q}\dot{p}\overline{\dot{q}}$.

We can think of a quaternion as an extended form of a complex number, whereby instead of two components, we have four. This is how mathematicians view quaternions — as an algebra over $\Re^4$.

The confounding quaternion multiplication can be represented in three equivalent ways:

Placeholder representation: We can represent $\dot{q}$ as $q_0 + q_x I + q_y J + q_z K$, and $\dot{p}$ as $p_0 + p_x I + p_y J + p_z K$. Then, we can obtain the formulas given above by defining the following operations between the I, J, and K placeholders:

$$\begin{array}{ccc}
II = -1 & JJ = -1 & KK = -1 \\
IJ = K & JK = I & KI = J \\
JI = -K & KJ = -I & IK = -J.
\end{array}$$

Vector representation: Representing $\dot{q}$ as $q + \vec{q}$, and $\dot{p}$ as $p + \vec{p}$, then $\dot{q}\dot{p} = (qp - \vec{q} \cdot \vec{p}) + (q\vec{p} + p\vec{q} + \vec{q} \times \vec{p})$. Note that this expression simplifies to $-\vec{q} \cdot \vec{p} + \vec{q} \times \vec{p}$ when $\dot{q}$ and $\dot{p}$ are purely imaginary ($q = p = 0$). We have used $\times$ to denote vector cross product.

Matrix representations: Define the matrix $Q_+(\dot{q})$ as

$$\begin{pmatrix}
q_0 & -q_x & -q_y & -q_z \\
q_x & q_0 & -q_z & q_y \\
q_y & q_z & q_0 & -q_x \\
q_z & -q_y & q_x & q_0
\end{pmatrix}.$$

We can then express $\dot{q}\dot{p}$ as $Q_+(\dot{q})\dot{p}$. In order to multiply a quaternion by a matrix, we first turn the quaternion into a column vector. Similarly, defining $Q_-(\dot{p})$ as

$$\begin{pmatrix}
p_0 & -p_x & -p_y & -p_z \\
p_x & p_0 & p_z & -p_y \\
p_y & -p_z & p_0 & p_x \\
p_z & p_y & -p_x & p_0
\end{pmatrix},$$

we obtain $\dot{q}\dot{p} = Q_-(\dot{p})\dot{q}$.

2.2 Useful Properties of Quaternions

There is a host of properties governing the quaternion algebra. We will only list some of those that are important. While the proofs are left to the reader, some guidance is provided. In most cases, proofs involve trivial and tedious algebraic manipulations.

1. $|\dot{q}|^2 = \dot{q} \cdot \dot{q} = \dot{q}\overline{\dot{q}}$.
2. The inverse exists provided $\dot{q} \neq 0$, and is equal to $\overline{\dot{q}}/|\dot{q}|^2$. This follows from property 1.
3. $\overline{\dot{q}\dot{p}} = \overline{\dot{p}}\,\overline{\dot{q}}$. The easiest way to prove this property is by using the vector representation of quaternion multiplication.
4. $(\dot{q}\dot{p})^{-1} = \dot{p}^{-1}\dot{q}^{-1}$, provided the inverses of $\dot{q}$ and $\dot{p}$ exist. This follows from properties 2 and 3.
5. $(\dot{q}\dot{p})\dot{r} = \dot{q}(\dot{p}\dot{r})$. In other words, quaternion multiplication is associative. But it is not commutative. The associativity property is equivalent to $Q_-(\dot{r})Q_+(\dot{q}) = Q_+(\dot{q})Q_-(\dot{r})$, which is easy to show.
6. $Q_+(\overline{\dot{q}}) = Q_+(\dot{q})^T$ and $Q_-(\overline{\dot{q}}) = Q_-(\dot{q})^T$, by inspection of the matrices involved; T denotes matrix transposition.
7. $C_{\dot{q}}(\dot{p}) = Q_-(\dot{q})^T Q_+(\dot{q})\dot{p}$. This follows from property 6.
8. $C_{\dot{q}}(\dot{p} + \dot{r}) = C_{\dot{q}}(\dot{p}) + C_{\dot{q}}(\dot{r})$ and $C_{\dot{q}}(s\dot{p}) = sC_{\dot{q}}(\dot{p})$, for real s. In other words, conjugation is a linear map. Both properties follow from property 7.
9. $C_{\dot{q}}(C_{\dot{p}}(\dot{r})) = C_{\dot{q}\dot{p}}(\dot{r})$. We can prove this by using property 3 and the definition of conjugation.

Property 7 is very important. It implies that, given a quaternion $\dot{q} = (a, b, c, d)$, then the mapping $C_{\dot{q}}(\cdot)$ is given by the matrix $\mathbf{R}_{\dot{q}} = Q_-(\dot{q})^T Q_+(\dot{q}) =$

$$\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 0 & 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 0 & 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

The sum of the squares of the entries in any column (or row) of $\mathbf{R}_{\dot{q}}$ is equal to $|\dot{q}|^4$. Furthermore, the dot product of any two columns (or rows) of $\mathbf{R}_{\dot{q}}$ is 0. Finally, the determinant of $\mathbf{R}_{\dot{q}}$ equals $|\dot{q}|^8$. Hence, if $|\dot{q}| = 1$, $\mathbf{R}_{\dot{q}}$ is a rotation matrix.

We have thus seen that a rotation can be represented by a unit quaternion! Since a quaternion comprises four independent components, and $|\dot{q}| = 1$ is a single constraint over the values of the components, it follows that a rotation has three d.o.f., as we have observed before.

In every term of every non-zero coefficient of $\mathbf{R}_{\dot{q}}$, the components of $\dot{q}$ appear in pairs, one component multiplying another. It follows that $\mathbf{R}_{\dot{q}} = \mathbf{R}_{-\dot{q}}$. Thus, $-\dot{q}$ and $\dot{q}$ represent the same rotation.

It is fairly straight forward to go in the other direction, i.e. given a rotation matrix $\mathbf{R}$, derive the quaternion $\dot{q}$ that represents the same rotation. Assume $\mathbf{R}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{00} & R_{01} & R_{02} \\ 0 & R_{10} & R_{11} & R_{12} \\ 0 & R_{20} & R_{21} & R_{22} \end{pmatrix}.$$

Then the following equations allow us to determine the magnitudes of the components of $\dot{q} = (a, b, c, d)$:

$$\begin{aligned} 1 + R_{00} + R_{11} + R_{22} &= 4a^2 \\ 1 + R_{00} - R_{11} - R_{22} &= 4b^2 \\ 1 - R_{00} + R_{11} - R_{22} &= 4c^2 \\ 1 - R_{00} - R_{11} + R_{22} &= 4d^2. \end{aligned}$$

The signs of a, b, c, and d can be derived from the following equations:

$$\begin{aligned} R_{21} - R_{12} &= 4ab \\ R_{02} - R_{20} &= 4ac \\ R_{10} - R_{01} &= 4ad \\ R_{10} + R_{01} &= 4bc \\ R_{21} + R_{12} &= 4cd \\ R_{02} + R_{20} &= 4bd. \end{aligned}$$

2.3 Why Bother

An advantage of quaternions over the matrix representation of rotations is that they allow us to deduce the axis and angle or rotation very easily from the quaternion components. If $\dot{q} = q + \vec{q}$ is a unit quaternion, then the (oriented) axis of rotation $\vec{\alpha}$ is given by

$$\frac{\vec{q}}{|\vec{q}\,|},$$

and the angle or rotation θ must satisfy the following two equations:

$$\sin\theta/2 = |\vec{q}\,| \quad \text{and} \quad \cos\theta/2 = q,$$

which imply

$$\sin\theta = 2q|\vec{q}\,| \quad \text{and} \quad \cos\theta = q^2 - |\vec{q}\,|^2.$$

As we said earlier, $-\dot{q}$ also represents the same rotation as $\dot{q}$. If so, we should obtain the same rotation axis and angle for $-\dot{q}$ as we did for $\dot{q}$. We do not. However, the rotation represented by $-\dot{q}$ is still the same as the one represented by $\dot{q}$. This "miracle" occurs because of a symmetry property of rotations: if we rotate around axis $\vec{\alpha}$ by angle θ, we effect the same transformation as if we rotate about $-\vec{\alpha}$ by $2\pi - \theta$. The quaternion $-\dot{q}$ yields the latter axis and angle, while $\dot{q}$ yields the former.

The inverse specification is easily derived. The quaternion representing a rotation about the axis $\vec{\alpha}$, a unit vector, by an angle θ is $\cos(\theta/2) + \vec{\alpha}\sin(\theta/2)$.

Another important advantage of quaternions over matrices is the efficiency of rotation composition when the rotations involved are represented by quaternions. Rotation composition is equivalent to quaternion multiplication, as we saw above (property 9), an operation which involves many fewer elementary arithmetic operations than 4×4 matrix multiplication.

[CP66] is very helpful for visualizing rotations and their Euler and quaternion representations. [KK68] provides a good summary of theorems and formulas for a variety of rotation representations. [FvDFH92] discusses the use of quaternions in animation.

References

[CP66] Brian H. Chirgwin and Charles Plumpton. *A Course of Mathematics for Engineers and Scientists*, volume 6. Pergamon, Oxford, NY, 1966.

[FvDFH92] James D. Foley, Andries van Dam, Steven K. Feiner, and John F. Hughes. *Computer Graphics: Principles and Practice*. Systems Programming Series. Addison, Reading, MA, second edition, 1992.

[KK68] Granino Arthur Korn and Theresa M. Korn. *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems, and Formulas for Reference and Review*. McGraw-Hill Handbooks. McGraw-Hill, New York, NY, second edition, 1968.