CS 348C **Fall 1995-6**
Assignment 1 Solutions **Apostolos Lerios**

Note

World Wide Web (WWW) documents referenced in these solutions can all be found under the course hierarchy

`http://graphics.stanford.edu/courses/cs348c-95-fall/`

Problem 1: Site Notation.

1. Let's first align the arbitrary tetrahedron T with the axes of an orthogonal coordinate system as shown in figure 1. This placement results in T's vertices having the following coordinates:

$$\begin{aligned} \mathbf{p} &= (0, 0, 0) \\ \mathbf{q} &= (a, 0, 0) \\ \mathbf{r} &= (u, b, 0) \\ \mathbf{s} &= (v, w, c) \end{aligned}$$

where a, b, c, u, v, and w depend on T's geometry.

A horizontal (i.e. parallel to the xy plane) cross-section of T at $z = z_0$ is shown in figure 2. The cross-section is a triangle with vertices $\mathbf{p}'$, $\mathbf{q}'$, and $\mathbf{r}'$: vertex $\mathbf{p}'$ of the triangle appears at the intersection of T's edge $\mathbf{ps}$ with the slicing plane; similarly, $\mathbf{q}'$ and $\mathbf{r}'$ lie respectively along T's edges $\mathbf{qs}$ and $\mathbf{rs}$.

$\mathbf{p}'$, $\mathbf{q}'$, and $\mathbf{r}'$ have all the same z coordinate z_0; their x and y coordinates can be found by linearly interpolating between the vertex positions for $z_0 = 0$ (when $\mathbf{p}'$, $\mathbf{q}'$, and $\mathbf{r}'$ overlap $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ respectively) and $z_0 = c$ (when $\mathbf{p}'$, $\mathbf{q}'$, and $\mathbf{r}'$ all coincide with $\mathbf{s}$). The interpolation weight is z_0/c since, as the slicing plane moves at constant speed from $z_0 = 0$ to $z_0 = c$, the ratio z_0/c steadily increases from 0 to 1.

Performing these interpolations yields the following coordinates for the triangle vertices:

$$\begin{aligned} \mathbf{p}' &= (\frac{z_0}{c}v, \frac{z_0}{c}w, z_0) \\ \mathbf{q}' &= (\frac{z_0}{c}v + (1 - \frac{z_0}{c})a, \frac{z_0}{c}w, z_0) \\ \mathbf{r}' &= (\frac{z_0}{c}v + (1 - \frac{z_0}{c})u, \frac{z_0}{c}w + (1 - \frac{z_0}{c})b, z_0). \end{aligned}$$

We can find the area $A(z_0)$ of this triangle from the formula discussed in class; we write $A(z_0)$ instead of just A, because the triangle vertices move

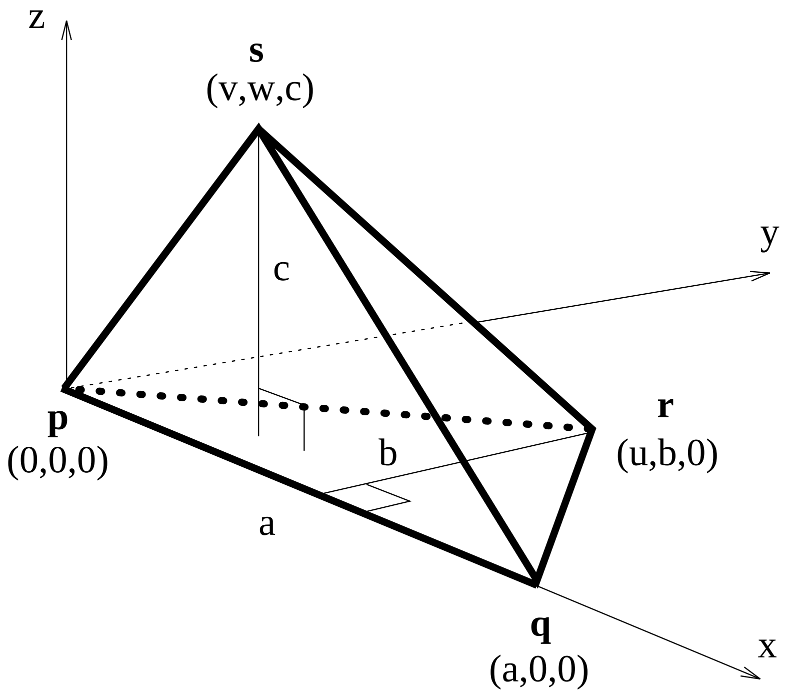

Figure 1: The sample tetrahedron of problem 1 placed in an orthogonal coordinate system.

closer together as the slicing plane moves upwards:

$$A(z_0) = \frac{1}{2}\begin{vmatrix} 1 & \frac{z_0}{c}v & \frac{z_0}{c}w \\ 1 & \frac{z_0}{c}v + (1-\frac{z_0}{c})a & \frac{z_0}{c}w \\ 1 & \frac{z_0}{c}v + (1-\frac{z_0}{c})u & \frac{z_0}{c}w + (1-\frac{z_0}{c})b \end{vmatrix}.$$

Subtracting the first row from the second and third rows simplifies this formula to

$$A(z_0) = \frac{1}{2}\begin{vmatrix} 1 & \frac{z_0}{c}v & \frac{z_0}{c}w \\ 0 & (1-\frac{z_0}{c})a & 0 \\ 0 & (1-\frac{z_0}{c})u & (1-\frac{z_0}{c})b \end{vmatrix}$$

which evaluates trivially to the product of the diagonal

$$A(z_0) = \frac{1}{2}(1-\frac{z_0}{c})^2 ab = \frac{ab}{2c^2}(c-z_0)^2.$$

In order to find the volume of the tetrahedron T, we first slice it up into infinitesimally thin triangles, and then add up the differential volumes of these triangles:

$$V = \int_0^c A(z)\,dz.$$

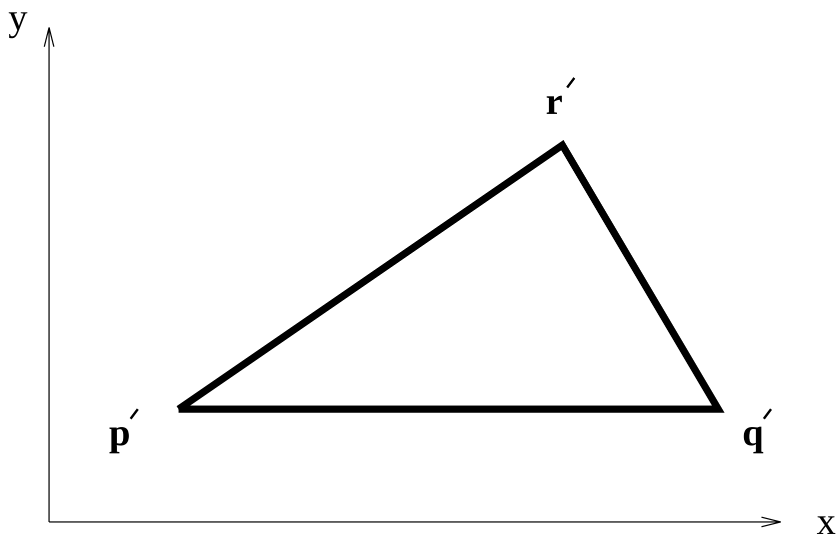

Figure 2: A horizontal slice of the tetrahedron of figure 1 at $z = z_0$.

This integral is easily evaluated, as the integrand is a simple polynomial:

$$\begin{aligned} V &= \int_0^c \frac{ab}{2c^2}(c-z)^2\,dz \\ &= \frac{ab}{2c^2}\int_0^c c^2 + z^2 - 2cz\,dz \\ &= \frac{ab}{2c^2}\left[c^2 z + \frac{z^3}{3} - cz^2\right]_0^c dz \\ &= \frac{ab}{2c^2}\left[c^3 + \frac{c^3}{3} - c^3\right]_0^c dz \\ &= \frac{abc}{6}. \end{aligned}$$

2. Let

$$\vec{n} = (\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p}).$$

This vector is shown in figure 3. Its magnitude is

$$|\vec{n}| = |\mathbf{q}-\mathbf{p}||\mathbf{r}-\mathbf{p}|\sin\theta = ab$$

and its direction is perpendicular to the plane $\mathbf{pqr}$.

We can apply the result above to simplify the following expression:

$$\begin{aligned} [(\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})] \cdot (\mathbf{s}-\mathbf{p}) &= |\vec{n}||\mathbf{s}-\mathbf{p}|\cos\phi \\ &= \pm abc. \end{aligned}$$

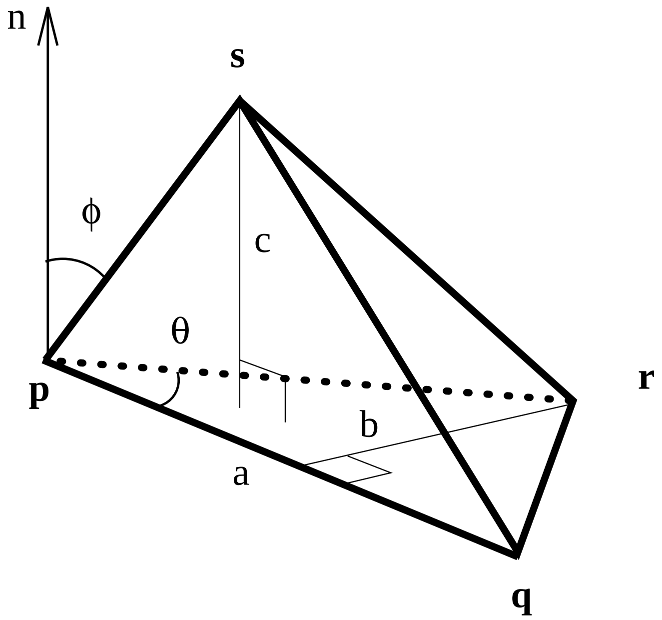

Figure 3: Expressing the volume of the tetrahedron of figure 1 in terms of vectors along its edges.

The sign of the result is positive if $\vec{n}$ is pointing from the plane $\mathbf{pqr}$ towards $\mathbf{s}$ because, in this case, $\phi < \pi/2$. Otherwise, it is negative (or zero).

From the first part and the last formula, we conclude that

$$\begin{aligned} V &= \frac{1}{6}abc \\ &= \frac{1}{6} \text{ abs } [((\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})) \cdot (\mathbf{s}-\mathbf{p})]. \end{aligned}$$

3. Using the Cartesian coordinates of T's vertices, we can find the components of the following vectors:

$$\begin{aligned} \mathbf{q}-\mathbf{p} &= (q_x - p_x, q_y - p_y, q_z - p_z) \\ \mathbf{r}-\mathbf{p} &= (r_x - p_x, r_y - p_y, r_z - p_z) \\ \mathbf{s}-\mathbf{p} &= (s_x - p_x, s_y - p_y, s_z - p_z). \end{aligned}$$

The cross product of the first two vectors can be expressed in determinant form, using the unit vectors along the coordinate axes $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$:

$$(\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ q_x - p_x & q_y - p_y & q_z - p_z \\ r_x - p_x & r_y - p_y & r_z - p_z \end{vmatrix}.$$

Taking the dot product with $\mathbf{s}-\mathbf{p}$ means that we multiply corresponding coordinates of the two vectors, and then add up the resulting products. That's exactly what the following expression achieves:

$$[(\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})] \cdot (\mathbf{s}-\mathbf{p}) = \begin{vmatrix} s_x - p_x & s_y - p_y & s_z - p_z \\ q_x - p_x & q_y - p_y & q_z - p_z \\ r_x - p_x & r_y - p_y & r_z - p_z \end{vmatrix}.$$

For example, substituting $s_x - p_x$ for $\vec{\imath}$ results in multiplying the x coordinates of the two dotted vectors.

Swapping the first and second rows of the above determinant flips its sign; but subsequently swapping the new second row with the third row restores the original sign, yielding

$$[(\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})] \cdot (\mathbf{s}-\mathbf{p}) = \begin{vmatrix} q_x - p_x & q_y - p_y & q_z - p_z \\ r_x - p_x & r_y - p_y & r_z - p_z \\ s_x - p_x & s_y - p_y & s_z - p_z \end{vmatrix}.$$

We can extend the above determinant into 4D by adding a row and a column as follows; expanding the determinant by the first column immediately yields the earlier form:

$$[(\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})] \cdot (\mathbf{s}-\mathbf{p}) = \begin{vmatrix} 1 & p_x & p_y & p_z \\ 0 & q_x - p_x & q_y - p_y & q_z - p_z \\ 0 & r_x - p_x & r_y - p_y & r_z - p_z \\ 0 & s_x - p_x & s_y - p_y & s_z - p_z \end{vmatrix}.$$

This expression simplifies to

$$[(\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})] \cdot (\mathbf{s}-\mathbf{p}) = \begin{vmatrix} 1 & p_x & p_y & p_z \\ 1 & q_x & q_y & q_z \\ 1 & r_x & r_y & r_z \\ 1 & s_x & s_y & s_z \end{vmatrix},$$

by adding the first row to all the others.

Finally, from the second part and the last formula, we conclude that

$$\begin{aligned} V &= \frac{1}{6} \operatorname{abs} \left[((\mathbf{q}-\mathbf{p}) \times (\mathbf{r}-\mathbf{p})) \cdot (\mathbf{s}-\mathbf{p}) \right] \\ &= \frac{1}{6} \operatorname{abs} \begin{vmatrix} 1 & p_x & p_y & p_z \\ 1 & q_x & q_y & q_z \\ 1 & r_x & r_y & r_z \\ 1 & s_x & s_y & s_z \end{vmatrix}. \end{aligned}$$

4. As we saw in the third part, the expression

$$D = \begin{vmatrix} 1 & p_x & p_y & p_z \\ 1 & q_x & q_y & q_z \\ 1 & r_x & r_y & r_z \\ 1 & s_x & s_y & s_z \end{vmatrix}$$

yields the same result as the expression

$$[(\mathbf{q}-\mathbf{p})\times(\mathbf{r}-\mathbf{p})]\cdot(\mathbf{s}-\mathbf{p})$$

which, as we saw in the second part, is positive if and only if $\vec{n}$ is pointing from the plane **pqr** towards **s**. But $\vec{n}$ is the result of a cross product. It follows that if

(a) I place by hand at **p**,

(b) align my fingers with **q**, and

(c) have my palm face **r**

— that's the standard way of finding the direction of a cross product —, then

- if my thumb points towards **s**, then $D > 0$,
- if my thumb points away from **s**, then $D < 0$, and
- if my index points directly towards **s**, i.e. if **s** lies on the plane **pqr**, then $D = 0$.

For the particular tetrahedron in the assignment, $D > 0$ since $\phi < \pi/2$; see figure 3.

Problem 2: Barycentric Coordinates.

1. By substituting the given points in the definition of barycentric coordinates we conclude that a_1, a_2, and a_3 must satisfy the following two equations:

$$\begin{aligned} 1 &= a_1 + a_2 + a_3 \\ (.5, .25) &= a_1(0,0) + a_2(0,1) + a_3(1,0). \end{aligned}$$

Splitting the second equation into its individual components (one for each coordinate), we get the following system of three equations in three unknowns:

$$\begin{aligned} 1 &= a_1 + a_2 + a_3 \\ .5 &= a_1 0 + a_2 0 + a_3 1 \\ .25 &= a_1 0 + a_2 1 + a_3 0. \end{aligned}$$

This system is trivially solvable, yielding

$$a_1 = .25 \quad a_2 = .25 \quad a_3 = .5.$$

2. Proceeding as before, we obtain the following three equations in three unknowns:

$$\begin{aligned} 1 &= a_1 + a_2 + a_3 \\ -6 &= a_1 1 + a_2 5 + a_3 6 \\ -6 &= a_1 2 + a_2 1 + a_3 4. \end{aligned}$$

   Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & 1 & \vdots & 1 \\ 0 & 1 & -2 & \vdots & 8 \\ 0 & 0 & 1 & \vdots & -3 \end{pmatrix},$$

   and back substitution produces the final result:

$$a_1 = 2 \quad a_2 = 2 \quad a_3 = -3.$$

3. Proceeding as usual, we obtain the following three equations in two unknowns:

$$\begin{aligned} 1 &= a_1 + a_2 \\ 5 &= a_1 2 + a_2 6 \\ 11 &= a_1 5 + a_2 13. \end{aligned}$$

   Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & \vdots & 1 \\ 0 & 1 & \vdots & .75 \\ 0 & 0 & \vdots & 0 \end{pmatrix},$$

   and back substitution produces the final result:

$$a_1 = .25 \quad a_2 = .75.$$

4. From the third part, we conclude that $\mathbf{p}_1 = (5, 11)$, $\mathbf{p}_2 = (2, 5)$ and $\mathbf{p}_3 = (6, 13)$ are collinear, otherwise, it would have been impossible to express $\mathbf{p}_1$ as a barycentric combination of $\mathbf{p}_2$ and $\mathbf{p}_3$. Hence, in this fourth part, we are asked to express $\mathbf{p} = (1, 3)$ as a barycentric combination of the three collinear points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. There can only be two answers to this problem:

   - $\mathbf{p}$ does not lie on the same line as $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. In this case, $\mathbf{p}$ cannot be expressed as a barycentric combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

- $\mathbf{p}$ lies on the same line as $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. In this case, there are infinite ways we can express $\mathbf{p}$ as a barycentric combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. For example, we may express $\mathbf{p}$ as a barycentric combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ (with $a_3 = 0$) or as a barycentric combination of $\mathbf{p}_2$ and $\mathbf{p}_3$ (with $a_1 = 0$).

To find out whether or not $\mathbf{p}$ lies on the same line as $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, let's attempt to express $\mathbf{p}$ as a barycentric combination of $\mathbf{p}_2$ and $\mathbf{p}_3$. Proceeding as before, we obtain the following three equations in two unknowns:

$$\begin{aligned} 1 &= a_2 + a_3 \\ 1 &= a_2 2 + a_3 6 \\ 3 &= a_2 5 + a_3 13. \end{aligned}$$

Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & \vdots & 1 \\ 0 & 1 & \vdots & -.25 \\ 0 & 0 & \vdots & 0 \end{pmatrix},$$

and back substitution produces a possible set of barycentric coordinates:

$$a_1 = 0 \quad a_2 = 1.25 \quad a_3 = -.25.$$

Since we were able to express $\mathbf{p}$ as a barycentric combination of $\mathbf{p}_2$ and $\mathbf{p}_3$, it follows that $\mathbf{p}$ lies on the same line as $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, and hence we can also express $\mathbf{p}$ as a barycentric combination of $\mathbf{p}_1$ and $\mathbf{p}_2$:

$$\begin{aligned} 1 &= a_1 + a_2 \\ 1 &= a_1 5 + a_2 2 \\ 3 &= a_1 11 + a_2 5. \end{aligned}$$

Gaussian elimination yields

$$\begin{pmatrix} 1 & 1 & \vdots & 1 \\ 0 & 1 & \vdots & \frac{4}{3} \\ 0 & 0 & \vdots & 0 \end{pmatrix},$$

and back substitution produces another possible result:

$$a_1 = -\tfrac{1}{3} \quad a_2 = \tfrac{4}{3} \quad a_3 = 0.$$

5. Proceeding as usual, we obtain the following three equations in two unknowns:

$$\begin{aligned} 1 &= a_1 + a_2 \\ 0 &= a_1 2 + a_2 6 \\ 0 &= a_1 5 + a_2 13. \end{aligned}$$

Gaussian elimination yields

$$\left(\begin{array}{cc:c} 1 & 1 & 1 \\ 0 & 1 & -.5 \\ 0 & 0 & -1 \end{array}\right),$$

which is not solvable. It follows that $\mathbf{p}$ cannot be expressed as a barycentric combination of $\mathbf{p}_1$ and $\mathbf{p}_2$.

6. From part 4 we know that $\mathbf{p}_1 = (1,3)$, $\mathbf{p}_2 = (2,5)$, and $\mathbf{p}_3 = (6,13)$ are collinear. From part 5, we know that $\mathbf{p} = (0,0)$ does not lie on the same line as $\mathbf{p}_2$ and $\mathbf{p}_3$. So, it cannot lie on the same line as $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. It follows that $\mathbf{p}$ cannot be expressed as a barycentric combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

Problem 3: Homogenous Coordinates.

1. In order to homogenize the line $x + y = 2$, we follow the definition of homogenization: for the projective point $(p_w\,; p_x\,, p_y)$ to lie on the projective curve, the corresponding Euclidean point, namely, $(p_x/p_w\,, p_y/p_w)$ should lie on the Euclidean curve, i.e.

$$\frac{p_x}{p_w} + \frac{p_y}{p_w} = 2$$

must hold. For this equation to properly define a projective curve, we must make sure that it is well-defined for points-at-infinity. To this end, we rewrite it as

$$p_x + p_y = 2p_w\,.$$

Proceeding as above for the line $x - y = 4$, yields the projective curve

$$p_x - p_y = 4p_w\,.$$

The intersection of these curves consists of all points $\mathbf{p} = (p_w\,; p_x\,, p_y)$ which lie on both curves. For $\mathbf{p}$ to lie on both curves, its projective coordinates must satisfy the equations of both curves. Hence it must be that

$$p_x = 2p_w - p_y \quad \text{and} \quad p_x = 4p_w + p_y\,.$$

It follows that

$$2p_w - p_y = 4p_w + p_y$$

which simplifies to

$$p_y = -p_w\,.$$

Letting c be an arbitrary *non-zero* real number, we conclude that any triple $(p_w\,; p_x, p_y)$ where

$$\begin{aligned} p_w &= c \\ p_y &= -c \\ p_x &= 2p_w - p_y = 3c \end{aligned}$$

satisfies the equations of both curves, and thus it represents a projective point which lies at the intersection of the given curves. Observe that all these triples represent the same projective point as they are mere multiples of each other.

2. Proceeding as before, homogenizing $3x+2y=4$ yields the projective curve

$$3p_x + 2p_y = 4p_w\,,$$

and homogenizing $-3x-2y=1$ yields

$$-3p_x - 2p_y = p_w\,.$$

For $(p_w\,; p_x, p_y)$ to lie on both curves, it it must be that

$$3p_x = 4p_w - 2p_y \quad \text{and} \quad 3p_x = -p_w - 2p_y\,.$$

It follows that

$$4p_w - 2p_y = -p_w - 2p_y$$

which can only be satisfied if $p_w = 0$. Letting c be an arbitrary *non-zero* real number, we conclude that any triple $(p_w\,; p_x, p_y)$ where

$$\begin{aligned} p_w &= 0 \\ p_y &= c \\ p_x &= -\frac{2}{3}p_y = -\frac{2}{3}c \end{aligned}$$

represents a projective point which lies at the intersection of the given curves. Observe that all these triples represent the same projective point-at-infinity, where the two given parallel (projective) lines intersect.

3. Proceeding as usual, homogenizing $xy=1$ yields the projective curve

$$p_x p_y = p_w^2\,,$$

and homogenizing $x=1$ yields

$$p_x = p_w\,,$$

In order to find $\mathbf{p} = (p_w\,; p_x, p_y)$ which lies on both curves, we substitute the second equation into the first yielding

$$p_w p_y = p_w^2\,.$$

Rewriting this last expression as

$$p_w(p_y - p_w) = 0,$$

we conclude that $\mathbf{p}$ lies on both curves iff either

$$\begin{aligned} p_w &= c \\ p_y &= c \\ p_x &= p_w = c, \end{aligned}$$

or

$$\begin{aligned} p_w &= 0 \\ p_y &= c \\ p_x &= p_w = 0, \end{aligned}$$

where c is an arbitrary *non-zero* real number. In words, we have deduced that the (projective) hyperbola $xy = 1$ and the (projective) vertical line $x = 1$ meet twice on the projective plane:

- once at the projective point $(c; c, c)$ whose corresponding Euclidean point is $(1, 1)$, and
- once at the projective point-at-infinity $(0; 0, c)$.

4. Proceeding as usual, homogenizing $xy = 1$ yields the projective curve

$$p_x p_y = p_w^2,$$

as in the previous part, and homogenizing $x = 0$ yields

$$p_x = 0,$$

In order to find $\mathbf{p} = (p_w; p_x, p_y)$ which lies on both curves, we substitute the second equation into the first one, obtaining

$$p_w^2 = 0. \tag{1}$$

We conclude that $\mathbf{p}$ lies on both curves iff

$$\begin{aligned} p_w &= 0 \\ p_y &= c \\ p_x &= 0, \end{aligned}$$

where c is an arbitrary *non-zero* real number. In words, we have deduced that the (projective) hyperbola $xy = 1$ and the (projective) vertical line $x = 0$ meet once on the projective plane at the projective point-at-infinity $(0; 0, c)$.

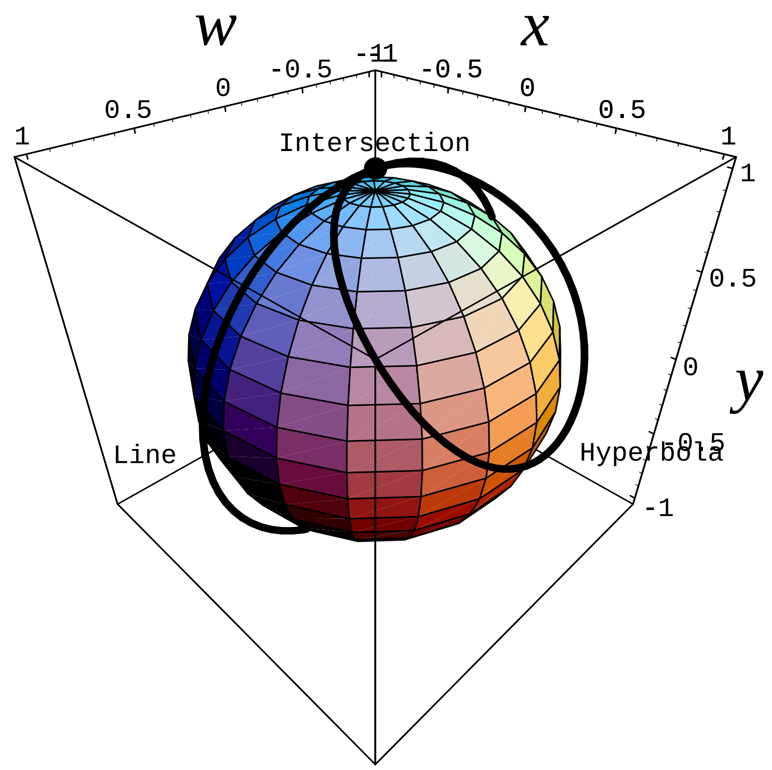

Figure 4: The curves of problem 3, part 5 as seen from the all positive quadrant.

In the previous part, we saw that the line $x = 1$ also meets the hyperbola at the same point-at-infinity. It follows that the parallel vertical (projective) lines $x = 0$ and $x = 1$ intersect at $(0; 0, c)$.

Also, it's interesting to observe that $p_w = 0$ is a double root of equation 1. As we'll see in the next part of the problem, this makes the (projective) line $x = 0$ *tangent* to the (projective) hyperbola $xy = 1$.

5. The desired 3D plot is shown from two different viewpoints in figures 4 and 5. The radius of the sphere shown is 0.9 to allow the curves to stand out. The three coordinate axes are labeled w, x, and y, representing the three components of the 3D point (w, x, y); this labeling was chosen because, as stated in the assignment, we plot the projective point $(p_w; p_x, p_y)$ at:

$$(p_w/l, p_x/l, p_y/l) \quad \text{and} \quad (-p_w/l, -p_x/l, -p_y/l)$$

where $l = \sqrt{p_w^2 + p_x^2 + p_y^2}$.

Drawing the projective line $p_x = 0$ is straightforward: we merely plot all points on the unit sphere whose second coordinate is zero.

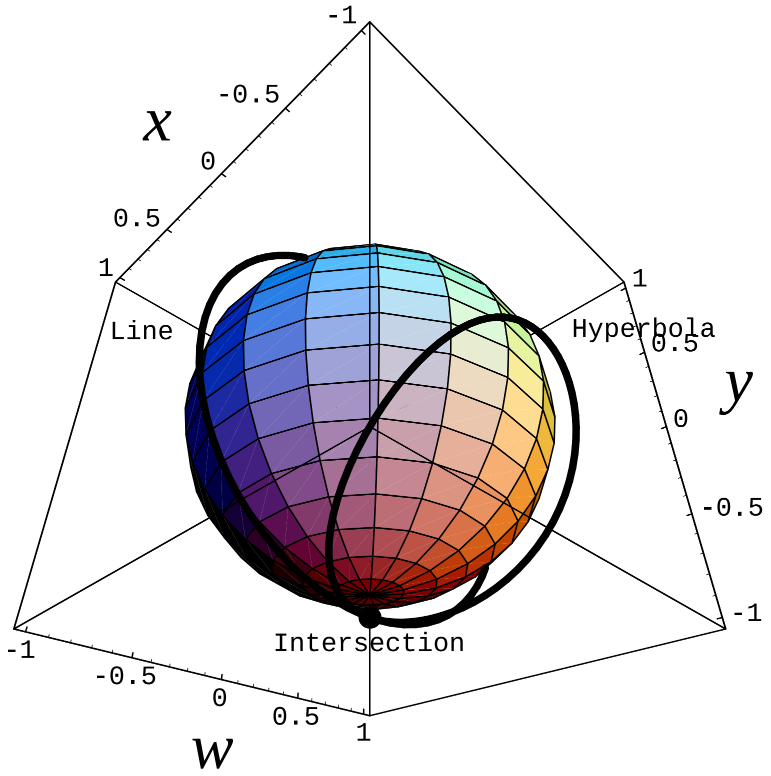

Figure 5: The curves of problem 3, part 5 as seen from the all negative quadrant.

There are several ways to go about drawing the hyperbola. Without using plotting software, you can think of $p_x p_y = p_w^2$ as a family of parabolas on the xw plane, with p_y defining the shape of each member of this family. Here are some sample members:

$$\begin{aligned}
p_y = 1 &\quad\mapsto\quad p_x = p_w^2 \\
p_y = 0.5 &\quad\mapsto\quad p_x = 2p_w^2 \\
p_y = 0.1 &\quad\mapsto\quad p_x = 10p_w^2 \\
p_y = -0.1 &\quad\mapsto\quad p_x = -10p_w^2 \\
p_y = -0.5 &\quad\mapsto\quad p_x = -2p_w^2 \\
p_y = -1 &\quad\mapsto\quad p_x = -p_w^2 .
\end{aligned}$$

As p_y decreases in absolute value from 1 to 0, the parabolas become sharper, i.e. increasing p_w at the same rate makes $|p_x|$ increase faster. In the limit, when $p_y = 0$, the parabola increases so fast that it becomes the x-axis itself ($p_w = 0$). To plot the hyperbola in 3D, we plot the intersections of these parabolas with the unit sphere. The parabolas for $p_y = \pm 1$ and the line for $p_y = 0$ intersect the unit sphere just once, while

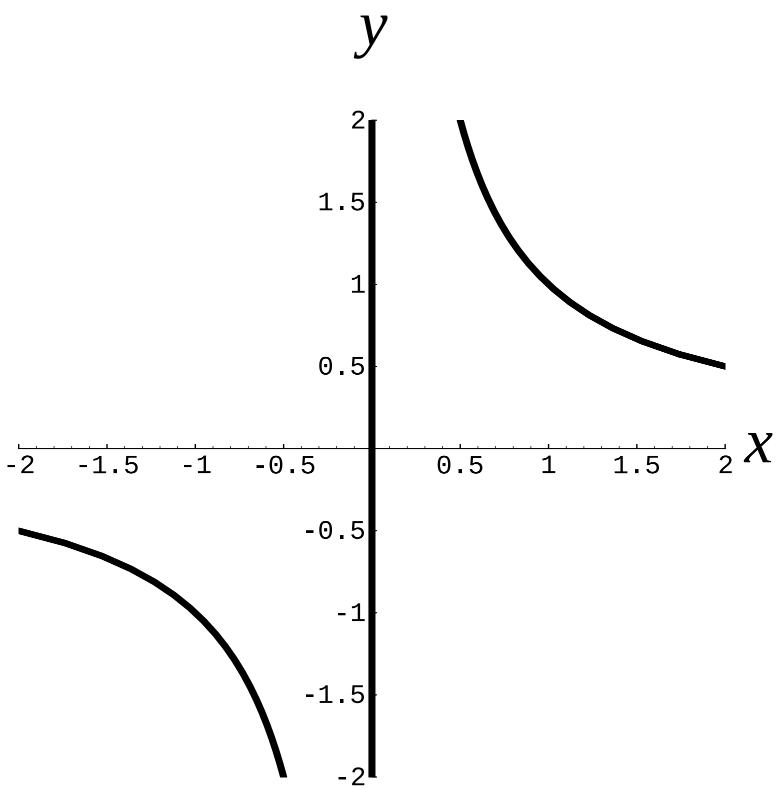

Figure 6: The Euclidean curves of problem 3, part 5.

all other parabolas intersect the sphere twice; these two points get closer together as p_y approaches ± 1 and 0.

Another way to go about drawing both curves is using Stolfi's construction relating the straight and spherical models of the projective plane. To this end, we first draw the Euclidean curves on a plane; see figure 6. Then, we place this plane in the 3D space; see figure 7. Finally, we project the Euclidean curves from the plane onto the unit sphere, yielding the desired visualization of the projective curves; see figures 8 and 9. See [Sto91, page 6] for more details.

In order to draw the hyperbola using `Mathematica`, we have to transform the implicit equation into a parametric one. That is, we presently describe the sets of 3D points we want to draw as the points (w, x, y) which satisfy the following two conditions:

- they lie on the unit sphere: $w^2 + x^2 + y^2 = 1$, and,
- they lie on the hyperbola: $xy = w^2$.

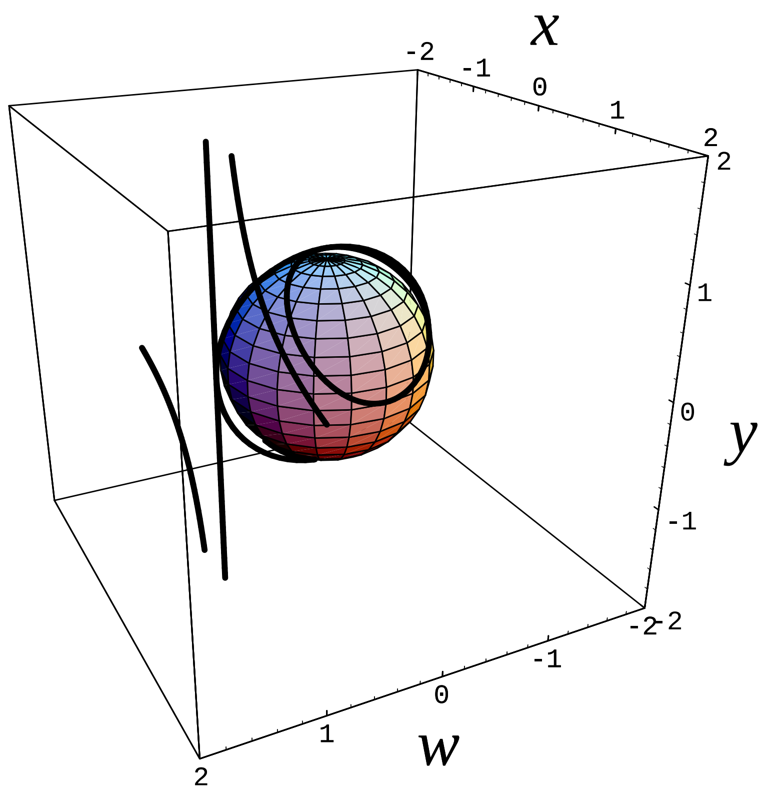

Figure 7: The Euclidean curves of problem 3, part 5 embedded in 3D space by placing the plane of figure 6 parallel to the xy plane at $w = 1$.

What **Mathematica** expects is a parametric curve in 3D. The first step in this transformation is to express every point on the unit sphere in terms of two parameters; the familiar parameterization of the sphere will do:

$$\begin{aligned} w &= \sin\phi\cos\theta \\ x &= \sin\phi\sin\theta \\ y &= \cos\phi. \end{aligned}$$

Next, we constrain θ and ϕ to yield points on the hyperbola:

$$(\sin\phi\sin\theta)(\cos\phi) = (\sin\phi\cos\theta)^2.$$

We then simplify this equation and rewrite it in two equivalent forms:

$$\begin{aligned} \cos\theta\cot\theta &= \cot\phi \\ \tan\theta\sec\theta &= \tan\phi. \end{aligned}$$

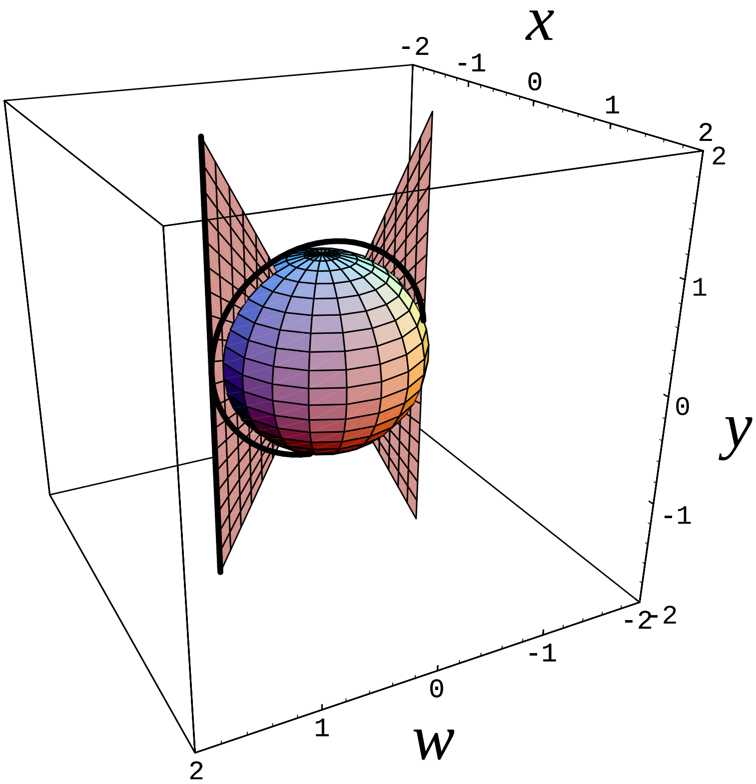

Figure 8: Projecting the Euclidean line of problem 3, part 5 from the embedded plane onto the unit sphere produces the desired drawing of the projective line. The projection rays are drawn as surfaces; we have only shown part of the Euclidean line and thus the corresponding projection does not fully comprise the projective line.

Finally, we use the two trigonometric identities

$$\begin{aligned} \sin\phi &= \frac{\pm 1}{\sqrt{1+\cot^2\phi}} \\ \cos\phi &= \frac{\pm 1}{\sqrt{1+\tan^2\phi}} \end{aligned}$$

to express the set of 3D points we want to draw as

$$\begin{aligned} w &= \frac{\pm\cos\theta}{\sqrt{1+\cos^2\theta\cot^2\theta}} \\ x &= \frac{\pm\sin\theta}{\sqrt{1+\cos^2\theta\cot^2\theta}} \\ y &= \frac{\pm 1}{\sqrt{1+\tan^2\theta\sec^2\theta}}. \end{aligned}$$

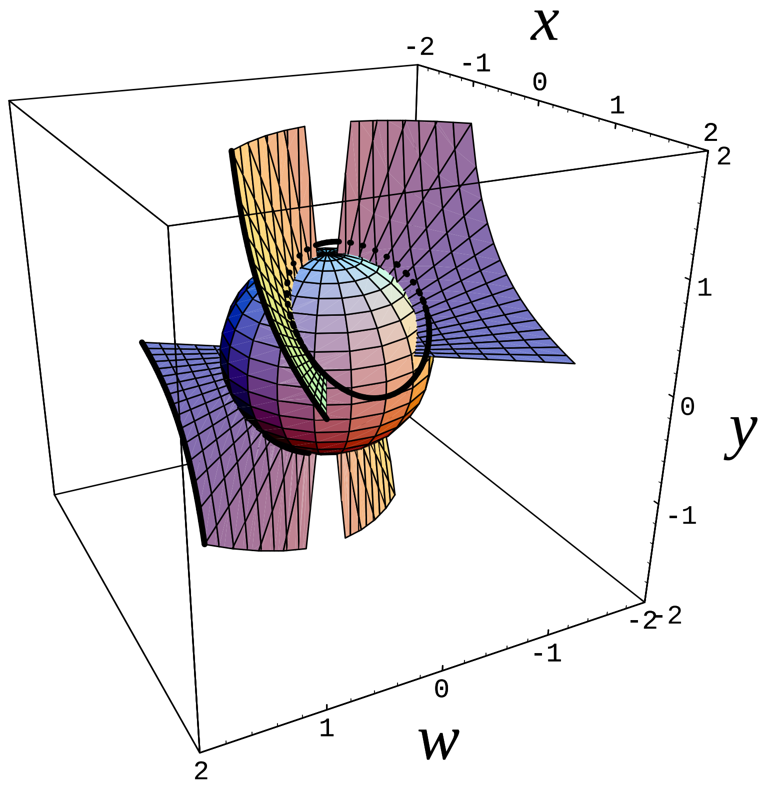

Figure 9: Projecting the Euclidean hyperbola of problem 3, part 5 from the embedded plane onto the unit sphere produces the desired drawing of the projective hyperbola. The projection rays are drawn as surfaces; we have only shown part of the Euclidean hyperbola and thus the corresponding projection does not fully comprise the projective hyperbola.

We can find the proper ranges for θ either by analyzing the above derivation, or by requiring that $xy = w^2$. It turns out that we can draw the curve in two segments, both of which use $\theta \in [0, \pi)$: the first segment uses the positive sign of the above parametric equations, while the second segment uses the negative sign.

Problem 4: Maps.

In order to solve this problem, we will have to resort to solving fairly complex linear systems. However, we can simplify these systems by observing that

$$\begin{pmatrix} i \\ e \\ f \end{pmatrix} = \begin{pmatrix} i & g & h \\ e & a & b \\ f & c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, the first column of $\mathbf{A}$ is trivially found from the point corresponding to $(1;0,0)$ under the given map.

1. Assuming a general projective map, and immediately using the fact that $(1;0,0)$ maps onto itself, we obtain

$$\mathbf{A} = \begin{pmatrix} 1 & g & h \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

To find the remaining coefficients of $\mathbf{A}$, we demand that it performs the desired map. So, for $(1;1,2)$ to map to $(1;4,11)$, we must have

$$\lambda \begin{pmatrix} 1 \\ 4 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & g & h \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The reason $\lambda \neq 0$ appears on the left-hand side is that all triples which are multiples of each other represent the same projective point. The above matrix equation reduces to the following three scalar equations:

$$\begin{aligned} \lambda &= 1 + g + 2h \\ 4\lambda &= a + 2b \\ 11\lambda &= c + 2d. \end{aligned}$$

Constructing similar constraints using the two other pairs of input and output points, we obtain the following system of nine equations in nine unknowns:

$$\begin{aligned} \lambda &= 1 + g + 2h \\ 4\lambda &= a + 2b \\ 11\lambda &= c + 2d \\ \mu &= 1 - g + 2h \\ 0 &= -a + 2b \\ 5\mu &= -c + 2d \\ \nu &= 1 - 3g - 6h \\ -12\nu &= -3a - 6b \\ -33\nu &= -3c - 6d. \end{aligned}$$

When `Mathematica` produces its solution, it expresses the result in terms of h; that is, the system is under-determined, possessing one degree of freedom. As linear maps are the most desirable, we set $h = 0$ to obtain

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & 4 \end{pmatrix}$$

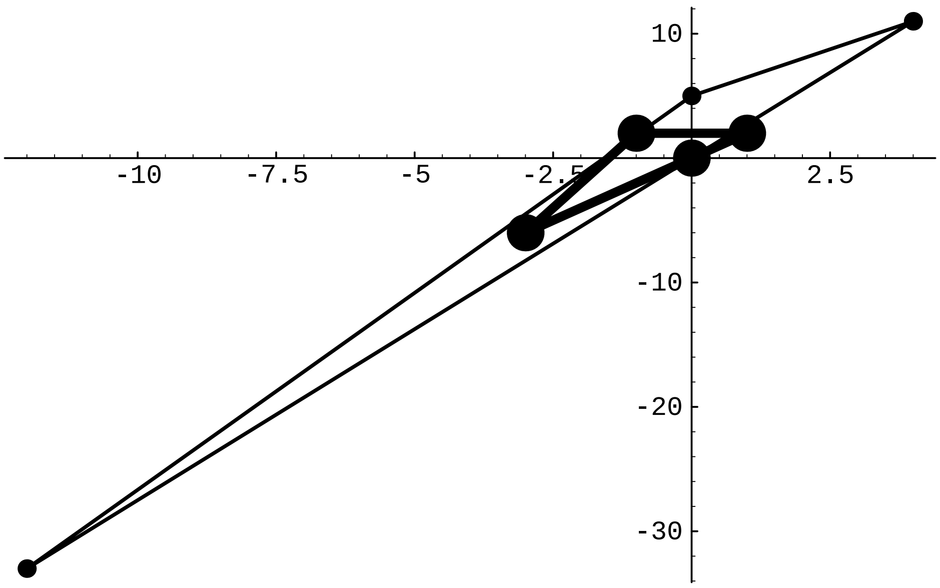

Figure 10: Visualization of the linear map of problem 4, part 1. The input points are drawn in bold.

with $\lambda = \mu = \nu = 1$.

It is not surprising that our system possessed one degree of freedom. Let's see why this is the case. Consider an arbitrary 3 by 3 matrix $\mathbf{B}$. This matrix defines a map on the projective plane by the relation

$$\mathbf{q} = \mathbf{B}\mathbf{p}.$$

However, non-zero multiples of the triples $(p_w\,; p_x, p_y)$ and $(q_w\,; q_x, q_y)$ all correspond to the same point; hence we can multiply both the left- and right-hand sides of the equation above with *two different* numbers and still define the same map. It follows that the nine numbers which comprise $\mathbf{B}$ afford us only *seven* degrees of freedom in defining an arbitrary projective map. By forcing the triple $(1; 0, 0)$ to map onto the same triple (instead of a multiple of itself), we exhausted one degree of freedom is our search for $\mathbf{A}$; the thus far unused second degree showed up when `Mathematica` solved the system.

Visualizing the map may give us some intuition on the effect of linear maps on planar objects. In figure 10 we have drawn the input and output points, as well as quadrilaterals whose vertices lie on the given points. We can see that a linear map does not change the basic shape of planar objects: while not immediately obvious, the four degrees of freedom of a linear map allow it to effect only rotations, scales and shears.

2. Proceeding as before, we assume a general projective map, and immediately use the fact that $(1;0,0)$ maps to $(1;5,4)$. We conclude that

$$\mathbf{A} = \begin{pmatrix} 1 & g & h \\ 5 & a & b \\ 4 & c & d \end{pmatrix}.$$

To find the remaining coefficients of $\mathbf{A}$, we demand that it performs the desired map. So we construct the following system of nine equations in nine unknowns:

$$\begin{aligned} \lambda &= 1 + g + 2h \\ 7\lambda &= 5 + a + 2b \\ 8\lambda &= 4 + c + 2d \\ \mu &= 1 - g + 2h \\ 3\mu &= 5 - a + 2b \\ 8\mu &= 4 - c + 2d \\ \nu &= 1 - 3g - 6h \\ -1\nu &= 5 - 3a - 6b \\ -8\nu &= 4 - 3c - 6d. \end{aligned}$$

When `Mathematica` produces its solution, it expresses the result in terms of μ. As affine maps are more desirable than projective ones, we set $\mu = 1$ so that $h = 0$ to obtain

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 4 & 0 & 2 \end{pmatrix}$$

with $\lambda = \nu = 1$.

The visualization of this affine map is shown in figure 11. We can see that an affine map, like a linear map, does not change the fundamental shape of planar objects: it turns out that affine maps extend the warping abilities of linear maps by adding only translations.

3. Proceeding as before, we assume a general projective map, and immediately use the fact that $(1;0,0)$ maps onto itself. We conclude that

$$\mathbf{A} = \begin{pmatrix} 1 & g & h \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

To find the remaining coefficients of $\mathbf{A}$, we demand that it performs the desired map. So we construct the following system of nine equations in nine unknowns:

$$\lambda = 1 + g + 2h$$

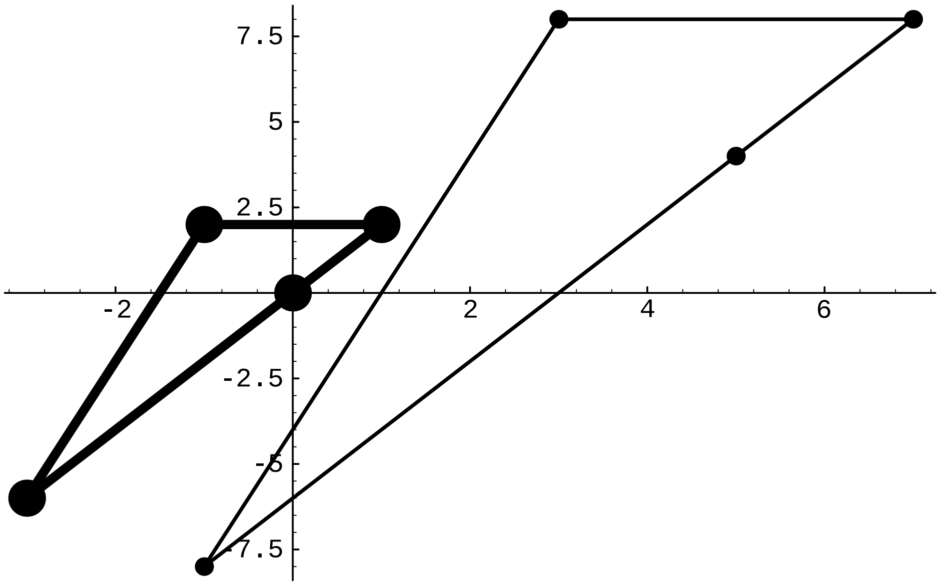

Figure 11: Visualization of the affine map of problem 4, part 2.

$$
\begin{aligned}
.25\lambda &= a + 2b \\
.5\lambda &= c + 2d \\
\mu &= 1 - g + 2h \\
-.5\mu &= -a + 2b \\
\mu &= -c + 2d \\
\nu &= 1 - 3g - 6h \\
.375\nu &= -3a - 6b \\
.75\nu &= -3c - 6d.
\end{aligned}
$$

When `Mathematica` produces its solution, it expresses the result in terms of μ. However, it is not possible to set μ so that $h = 0$; instead, we pick $\mu = 2$ so that the result has a simple form:

$$
\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

with $\lambda = 4$, $\nu = -8$.

The visualization of this projective map is shown in figure 12. We can see that a projective map can substantially alter the shape of planar objects: a projective map can map any quadrilateral (defined by the four input points) to any other (defined by the four output points).

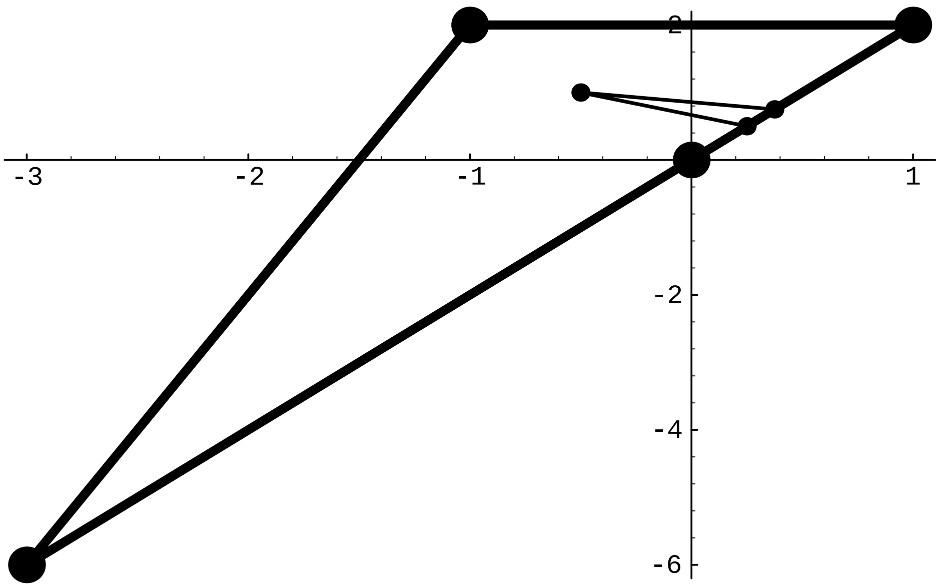

Figure 12: Visualization of the projective map of problem 4, part 3.

Problem 5: Parametric Polynomial Curves.

Programming part.

A sufficiently efficient solution to this part is shown below:

```
static int PointCount;
static Point *ControlP;


/* Initialization: retain basic system parameters in local
   variables. */

void InitSystem(int NewPointCount,
                Point *NewControlP) {

  PointCount=NewPointCount;
  ControlP=NewControlP;
}

/* Point retrieval. */

void GetCurvePoint(int Bezier,
                   double t,
```

```
                    Point P) {

  int i,j,k;
  double dt;
  double Basis;

  /* Minor optimization: dt is the difference between parameter values
     at interpolated points. */

  dt=1.0/(PointCount-1);

  /* Clear result. */

  for (i=0;i<3;i++)
    P[i]=0.0;

  /* Consider every control point in turn. */

  for (i=0;i<PointCount;i++) {

    /* Evaluate basis function for current control point at t. */

    if (Bezier) {
      Basis=pow(t,i)*pow(1.0-t,PointCount-1-i);
      for (j=i+1,k=1;j<PointCount;j++,k++)
        Basis*=((double)j)/k;
    } else {
      Basis=1.0;
      for (j=0;j<PointCount;j++)
        if (i!=j)
          Basis*=(t-dt*j)/((i-j)*dt);
    }

    /* Add contribution of control point to result. */

    for (j=0;j<3;j++)
      P[j]+=ControlP[i][j]*Basis;
  }
}

/* Cleanup. */

void Cleanup(void) {

}
```

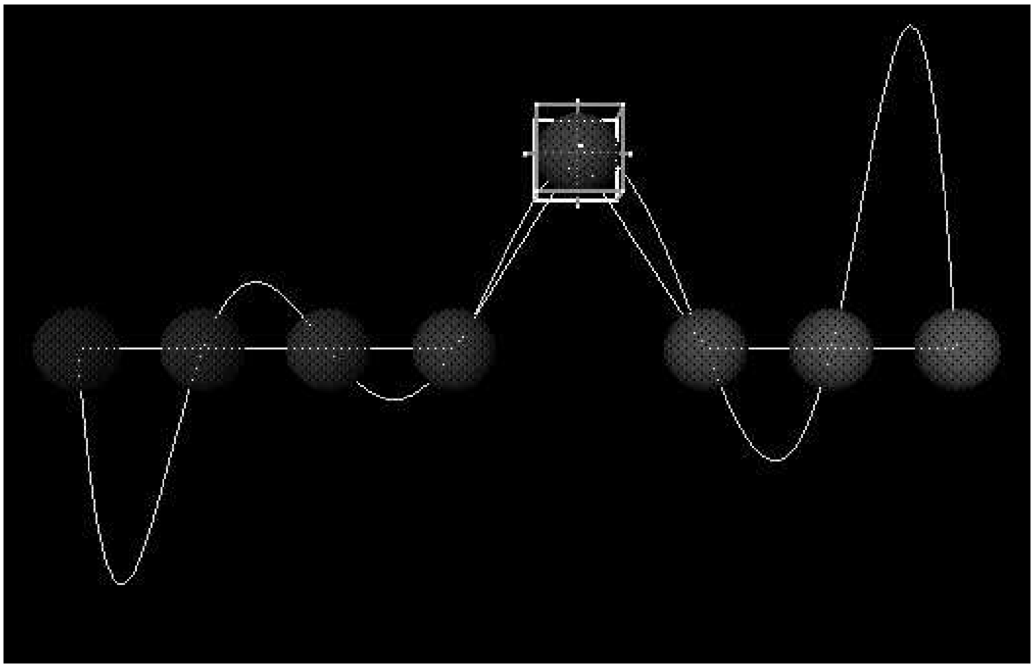

Figure 13: The output of `polly` for the interpolating curve described in part 1 of problem 5.

If you want to try it out, the source code of `polly` is now available on-line as a `tar` (archive) file; see the WWW page

`software/polly/index.html`

Written questions.

1. Figure 13 shows the output of `polly` for an interpolating curve fitting the description of the assignment; this curve has 3 "ups" and 4 "downs". Figure 14 shows the output of `polly` for the required Bézier curve; this curve has 1 "up" and 1 "down".

2. Animating the movement of a particle produces the following results:

   - Using the command-line

     ```
     polly
     ```

     the particle moves with constant speed from the first control point to the last.

   - Using the command-line

     ```
     polly -points 0 0 0 6 0 0 3 0 0 9 0 0
     ```

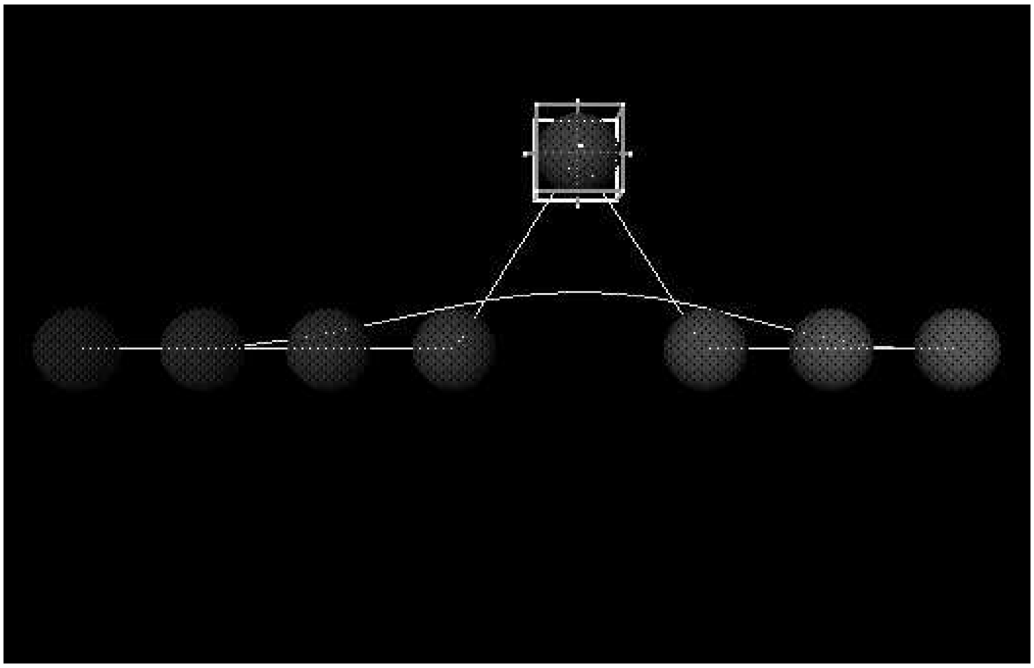

Figure 14: The output of `polly` for the Bézier curve described in part 1 of problem 5.

the particle moves from the first control point to the second one (which corresponds to the *third* sphere from the left). Then, the particle changes direction and moves backwards to the third control point (the *second* sphere from the left). Finally, it changes direction again, and moves to the right until it encounters the fourth control point.

References

[Sto91] Jorge Stolfi. *Oriented Projective Geometry: A Framework for Geometric Computations.* Academic P., Boston, MA, 1991.