Overview

Earlier lecture
- Statistical sampling and Monte Carlo integration

Last lecture
- Signal processing view of sampling

Today
- Variance reduction
- Importance sampling
- Stratified sampling
- Multidimensional sampling patterns
- Discrepancy and Quasi-Monte Carlo

Latter
- Path tracing for interreflection
- Density estimation

Cameras

\[ R = \int_{T} \int_{\Omega} \int_{A} L(x, \omega, t) P(x) S(t) \cos \theta dA d\omega dt \]

Motion Blur

Depth of Field

Source: Cook, Porter, Carpenter, 1984

Source: Mitchell, 1991
Variance

Definition


Variance decreases with sample size

\[ V\left[ \frac{1}{N} \sum_{i=1}^{N} Y_i \right] = \frac{1}{N} V[Y] \]
Examples

Projected solid angle

- 4 eye rays per pixel
- 100 shadow rays

Area

- 4 eye rays per pixel
- 100 shadow rays

Variance Reduction

Efficiency measure

\[ \text{Efficiency} \propto \frac{1}{\text{Variance} \cdot \text{Cost}} \]

Some techniques

- Estimators
- Expected values vs. rejection sampling
- Importance sampling
- Sampling patterns: stratified, correlated, antithetic
Biasing

Biasing the sampling process

\[ X_i \sim p(x) \quad Y_i = \frac{f(X_i)}{p(X_i)} \]

\[ E[Y_i] = E \left[ \frac{f(X_i)}{p(X_i)} \right] \]
\[ = \int \frac{f(X_i)}{p(X_i)} p(x) \, dx \]
\[ = \int f(x) \, dx \]
\[ = I \]

Importance Sampling

Variance

\[ V[f] = E[f^2] - E^2[f] \quad E[Y_i^2] = \int \left[ \frac{f(X_i)}{p(X_i)} \right]^2 p(x) \, dx \]

Zero variance biasing

\[ \tilde{p}(x) = \frac{f(x)}{E[f]} \]
\[ E[\tilde{f}^2] = \int \left[ \frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x) \, dx \]
\[ = \int \left[ \frac{f(x)}{f(x) / E[f]} \right]^2 \frac{f(x)}{E[f]} \, dx \]
\[ = E^2[f] \]
Irradiance

Generate cosine weighted distribution

\[ p(\omega) \, d\omega = \cos \theta \, d\omega \]

\[ E = \int_{H^2} L_i(\omega_i) \cos \theta_i \, d\omega_i \]

Stratified Sampling

Stratified sampling like jittered sampling

Allocate samples per region

\[ N = \sum_{i=1}^{m} N_i \quad F_N = \frac{1}{N} \sum_{i=1}^{m} N_i F_i \]

New variance

\[ V[F_N] = \frac{1}{N^2} \sum_{i=1}^{N} N_i V[F_i] \]

Thus, if the variance in regions is less than the overall variance, there will be a reduction in resulting variance

For example: An edge through a pixel

\[ V[F_N] = \frac{1}{N^2} \sum_{i=1}^{m} N_i V[F_i] = \frac{V[F_E]}{N^{1.5}} \]
High-dimensional Sampling

Stratified sampling (also numerical quadrature)
For a given error ...

\[ E \sim \frac{1}{N^{d/2}} \]

Random sampling
For a given variance ...

\[ E \sim N^{1/2} \sim \frac{1}{N^{1/2}} \]

Monte Carlo much better for
Integration in high dimensional spaces

Shirley’s Mapping

\[ r = U_1 \]
\[ \theta = \frac{\pi}{4} \frac{U_2}{U_1} \]
Block Design

<table>
<thead>
<tr>
<th>a</th>
<th>d</th>
<th>c</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>a</td>
<td>d</td>
<td>c</td>
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</tr>
<tr>
<td>d</td>
<td>c</td>
<td>b</td>
<td>a</td>
</tr>
</tbody>
</table>

Alphabet of size $n$
Each symbol appears exactly once in each row and column
Improves discrepancy

Latin Square

<table>
<thead>
<tr>
<th>a</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
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<tr>
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</table>

Incomplete block design

Replaced $N^d$ samples with $N$ samples

Permutations: $(\pi_1(i), \pi_2(i), \cdots \pi_d(i))$

Generalizations: N-queens, 2D projection

$(\pi_x = \{1, 2, 3, 4\}, \pi_y = \{4, 2, 3, 1\})$

Space-time Patterns

<table>
<thead>
<tr>
<th>6</th>
<th>10</th>
<th>2</th>
<th>13</th>
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<tr>
<td>3</td>
<td>14</td>
<td>12</td>
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<td>7</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>4</td>
<td>1</td>
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</tbody>
</table>

Distribute $t$ samples

- Decorrelate space and time
- Nearby samples in space should differ greatly in time

Cook Pattern

<table>
<thead>
<tr>
<th>15</th>
<th>8</th>
<th>5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>14</td>
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</tr>
<tr>
<td>1</td>
<td>6</td>
<td>11</td>
<td>12</td>
</tr>
</tbody>
</table>

Pan-diagonal Magic Square
Examples

4 eye rays per pixel
16 shadow rays per eye ray

64 eye rays per pixel
1 shadow ray per eye ray

Uniform random
Spectrally optimized
Discrepancy

\[ \Delta(x, y) = \frac{n(x, y)}{N} - xy \]

\[ A = xy \]

\[ n(x, y) \text{ number of samples in } A \]

\[ D_N = \max_{x,y} |\Delta(x, y)| \]

Theorem on Total Variation

**Theorem:**

\[ \left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \int f(x) \, dx \right| \leq V(f)D_N \]

**Proof:** Integrate by parts

\[ \int f(x) \left[ \frac{\delta(x-x_i)}{N} - 1 \right] \, dx \]

\[ = \int f(x) \frac{\partial \Delta(x)}{\partial x} \, dx \]

\[ = f\Delta_0' - \int \frac{\partial f(x)}{\partial x} \Delta(x) \ldots \, dx = -\int \frac{\partial f(x)}{\partial x} \Delta(x) \, dx \]

\[ \leq D_N \int \left| \frac{\partial f(x)}{\partial x} \right| \, dx = V(f)D_N \]
Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)

of integer $i$ in integer base $b$

$$i = d_i \ldots d_2 d_1 d_0$$

$$\phi_b(i) \equiv 0.d_i d_{i-1} \ldots d_1$$

<table>
<thead>
<tr>
<th>$b$</th>
<th>$\phi_2(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.1</td>
</tr>
<tr>
<td>2</td>
<td>10.01</td>
</tr>
<tr>
<td>3</td>
<td>11.11</td>
</tr>
<tr>
<td>4</td>
<td>100.001</td>
</tr>
<tr>
<td>5</td>
<td>101.01</td>
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</tbody>
</table>

Hammersley points

$$\frac{i}{N}, \phi_2(i), \phi_3(i), \phi_4(i), \ldots$$

$$D_N = O\left(\frac{\log^{d-1} N}{N}\right)$$

Halton points (sequential)

$$\phi_2(i), \phi_3(i), \phi_4(i), \ldots$$

$$D_N = O\left(\frac{\log^d N}{N}\right)$$

Edge Discrepancy

Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1,2,4,8 samples
Low-Discrepancy Patterns

<table>
<thead>
<tr>
<th>Process</th>
<th>16 points</th>
<th>256 points</th>
<th>1600 points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zaremba</td>
<td>0.0504</td>
<td>0.00478</td>
<td>0.00111</td>
</tr>
<tr>
<td>Jittered</td>
<td>0.0538</td>
<td>0.00595</td>
<td>0.00146</td>
</tr>
<tr>
<td>Poisson-Disk</td>
<td>0.0613</td>
<td>0.00767</td>
<td>0.00241</td>
</tr>
<tr>
<td>N-Rooks</td>
<td>0.0637</td>
<td>0.0123</td>
<td>0.00488</td>
</tr>
<tr>
<td>Random</td>
<td>0.0924</td>
<td>0.0224</td>
<td>0.00866</td>
</tr>
</tbody>
</table>

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended

Views of Integration

1. Signal processing
   - Sampling and reconstruction, aliasing and antialiasing
   - Blue noise good
2. Statistical sampling
   - Monte Carlo: variance, central limit theorem
   - Adaptive sampling criteria
   - $N^{-1/2}$: high dimensional sampling
3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions
4. Numerical
   - Quadrature/Integration rules
   - Smooth functions