Overview

Last Lecture
- Statistical sampling and Monte Carlo integration

Today
- Variance reduction
- Importance sampling
- Discrepancy and Quasi-Monte Carlo
- Multidimensional sampling patterns

Latter
- Path tracing for interreflection
- Density estimation

Variance Reduction

Efficiency measure

\[ \text{Efficiency} \propto \frac{1}{\text{Variance} \cdot \text{Cost}} \]

Some techniques
- Estimators
- Expected values vs. rejection sampling
- Importance sampling
- Sampling patterns: stratified, correlated, antithetic
**Biasing**

**Biasing the sampling process**

\[ X_i \sim p(x) \quad Y_i = \frac{f(X_i)}{p(X_i)} \]

\[ E[Y_i] = E \left[ \frac{f(X_i)}{p(X_i)} \right] \]

\[ = \int \frac{f(X_i)}{p(X_i)} p(x) \, dx \]

\[ = \int f(x) \, dx \]

\[ = I \]

---

**Importance Sampling**

**Variance**

\[ V[f] = E[f^2] - E^2[f] \]

\[ E[Y_i^2] = \int \left[ \frac{f(X_i)}{p(X_i)} \right]^2 p(x) \, dx \]

**Zero variance biasing**

\[ \tilde{p}(x) = \frac{f(x)}{E[f]} \]

\[ E[\tilde{f}^2] = \int \left[ \frac{f(x)}{\tilde{p}(x)} \right]^2 \tilde{p}(x) \, dx \]

\[ = \int \left[ \frac{f(x)}{f(x) / E[f]} \right]^2 \frac{f(x)}{E[f]} \, dx \]

\[ = E^2[f] \]
Irradiance

Generate cosine weighted distribution

\[ p(\omega) d\omega = \cos \theta \, d\omega \]

\[ E = \int_{H^2} L_i(\omega_i) \cos \theta_i \, d\omega_i \]
Multiple Importance Sampling

Reflection of a circular light source by a rough surface

\[ \int f(x)g(x) \, dx \]

<table>
<thead>
<tr>
<th>Radius</th>
<th>Shininess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sampling the light source</td>
<td>Sampling the BRDF</td>
</tr>
</tbody>
</table>

Two sampling techniques

\[
X_{1,i} \sim p_1(x) \quad X_{2,i} \sim p_2(x)
\]

\[
Y_{1,i} = \frac{f(X_{1,i})}{p_1(X_{1,i})} \quad Y_{2,i} = \frac{f(X_{2,i})}{p_2(X_{2,i})}
\]

Form weighted combination of samples

\[
Y_i = w_1 Y_{1,i} + w_2 Y_{2,i}
\]

The balance heuristic

\[
w_i(x) = \frac{p_i(x)}{p_1(x) + p_2(x)} \Rightarrow p(x) = w_1(x)p_1(x) + w_2(x)p_2(x)
\]
Multiple Importance Sampling

Combine both sampling methods

Source: Veach and Guibas

Discrepancy

\[ \Delta(x, y) = \frac{n(x, y)}{N} - xy \]

\[ A = xy \]

\[ n(x, y) \text{ number of samples in } A \]

\[ D_N = \max_{x,y} |\Delta(x, y)| \]
Edge Discrepancy

Note: SGI IR Multisampling extension:
8x8 subpixel grid; 1, 2, 4, 8 samples

Quasi-Monte Carlo Patterns

Radical inverse (digit reverse)
of integer \(i\) in integer base \(b\)

\[
i = d_1 \cdots d_2 d_0
\]

\[
\phi_b(i) \equiv 0.d_0d_1d_2 \cdots d_i
\]

Hammersley points

\[
(i/N, \phi_2(i), \phi_3(i), \phi_4(i), \cdots)
\]

Halton points (sequential)

\[
(\phi_2(i), \phi_3(i), \phi_4(i), \cdots)
\]

\[
D_N = O\left(\frac{\log^{d-1} N}{N}\right)
\]

\[
D_N = O\left(\frac{\log^d N}{N}\right)
\]
Low-Discrepancy Patterns

<table>
<thead>
<tr>
<th>Process</th>
<th>16 points</th>
<th>256 points</th>
<th>1600 points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zaremba</td>
<td>0.0504</td>
<td>0.00478</td>
<td>0.00111</td>
</tr>
<tr>
<td>Jittered</td>
<td>0.0538</td>
<td>0.00595</td>
<td>0.00146</td>
</tr>
<tr>
<td>Poisson-Disk</td>
<td>0.0613</td>
<td>0.00767</td>
<td>0.00241</td>
</tr>
<tr>
<td>N-Rooks</td>
<td>0.0637</td>
<td>0.0123</td>
<td>0.00488</td>
</tr>
<tr>
<td>Random</td>
<td>0.0924</td>
<td>0.0224</td>
<td>0.00866</td>
</tr>
</tbody>
</table>

Discrepancy of random edges, From Mitchell (1992)

Random sampling converges as $N^{-1/2}$
Zaremba converges faster and has lower discrepancy
Zaremba has a relatively poor blue noise spectra
Jittered and Poisson-Disk recommended
Story still unclear in higher dimensions?

Theorem on Total Variation

**Theorem:**

$$ \left| \frac{1}{N} \sum_{i=1}^{N} f(X_i) - \int f(x) \, dx \right| \leq V(f) D_N $$

**Proof:** Integrate by parts

$$ \int f(x) \left[ \frac{\delta(x-x_i)}{N} - 1 \right] \, dx = \int f(x) \frac{\partial \delta(x)}{\partial x} \, dx $$

$$ = \int f(x) \frac{\partial \Delta(x)}{\partial x} \, dx $$

$$ = f \Delta_{[1]}^i - \int \frac{\partial f(x)}{\partial x} \Delta(x) \cdots \, dx = -\int \frac{\partial f(x)}{\partial x} \Delta(x) \, dx $$

$$ \leq D_N \int \left| \frac{\partial f(x)}{\partial x} \right| \, dx = V(f) D_N $$
Block Design

- Alphabet of size \( n \)
- Each symbol appears exactly once in each row and column
- Improves discrepancy

<table>
<thead>
<tr>
<th>a</th>
<th>d</th>
<th>c</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>b</td>
<td>a</td>
<td>d</td>
<td>c</td>
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<tr>
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<td>a</td>
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</table>

Latin Square

Incomplete block design

- Replaced \( N^d \) samples with \( N \) samples

Permutations: \((\pi_x(i), \pi_y(i), \cdots \pi_d(i))\)

Generalizations: N-queens, 2D projection

\((\pi_x = \{1, 2, 3, 4\}, \pi_y = \{4, 2, 3, 1\})\)

N-Rook Pattern

Space-time Patterns

- Fully populate \((x, y)\) samples
  - Recall blue noise good
    - Perceptually pleasing
    - Filtered during resampling
  - Jitter to achieve blue noise

- Distribute \( t \) samples
  - Decorrelate space and time
  - Nearby samples in space should differ greatly in time

Mitchell (1991) designs
Views of Integration

1. Signal processing
   - Sampling and reconstruction, aliasing and antialiasing
   - Blue noise good

2. Statistical sampling
   - Monte Carlo: variance, central limit theorem
   - Adaptive sampling criteria, $N^{-1/2}$, high dimensional sampling

3. Quasi Monte Carlo
   - Discrepancy
   - Asymptotic efficiency in high dimensions

4. Numerical
   - Quadrature/Integration rules
   - Smooth functions