Homework #1: Point coordinates and line coefficients; affine and projective geometry; transformations and quaternions [65 points]
Due Date: Tuesday, 26 January 1999

Homework policies

CS348a is a highly technical course, so doing the homework is the only way to acquire a working knowledge of the material presented. We encourage you strongly to start working on the homework problems right away—the problems below, as well as those to follow, have considerable technical depth and you are unlikely to be able to solve them if you wait until the evening before the due date.

Collaboration in solving the problems is encouraged in this class—you have a lot to learn from your fellow students. However, in order to make grading the homeworks a meaningful way to measure your effort and your understanding of the material, we must put some restrictions:

- *On theoretical (mathematical) problems, you may work together on finding solutions, but each of you must then write up your favorite solutions independently. Please list the names of your collaborators on your homework.*
- *On programming problems, groups of up to three students can work together as a team, handing in a single body of code and documentation as their team effort.*

It is very important in this course that every homework be turned in on time. We recognize that occasionally there are circumstances beyond your control that prevent an assignment from being completed on time. You will be allowed two classes of grace during the quarter. This means that you can either hand in two assignments each late by one class, or one assignment late by two classes. Any further assignments handed in late will be penalized by 20% for each class that they are late, unless special arrangements have been made previously with the instructor or the TA.

Problem 1. [15 points]

Consider the parabola $Y = X^2$ in the plane. As the real number t varies, the point $B(t) := (1; t, t^2)$ traces out that parabola. Assuming that p and q are distinct real numbers, find the homogeneous coefficients of the chord ℓ_{pq}, the line that joins the point $B(p)$ to the point $B(q)$. By letting p and q both approach a common value t, find the homogeneous coefficients of the tangent line ℓ_{tt} to the parabola B at the point $B(t)$. Find the (non-homogeneous) coordinates of the velocity vector $B'(t)$, and verify that the tangent line ℓ_{tt} contains both the point $B(t)$ and the vector $B'(t)$. Find the (non-homogeneous) coordinates of the point B_{pq} where the tangent lines ℓ_{pp} and ℓ_{qq} intersect. Show that, as long as p and q are distinct, the point B_{pq} never lies on the line ℓ_{pq}.

In a similar way, consider the twisted cubic curve in 3-space traced out by the varying point $C(t) := (1; t, t^2, t^3)$. Assuming that p, q, and r are distinct, find the homogeneous coefficients of the plane π_{pqr} that passes through the points $C(p)$, $C(q)$, and $C(r)$. By letting p, q, and r all approach t, find the homogeneous coefficients of the *osculating plane* π_{ttt}, the plane that most nearly contains the curve C in the neighborhood of the point $C(t)$. Find the coordinates of the velocity vector $C'(t)$ and of the acceleration vector $C''(t)$, and verify that the osculating plane π_{ttt} contains the three sites $C(t)$, $C'(t)$, and $C''(t)$. Find the coordinates of the point C_{pqr} where the three osculating planes π_{ppp}, π_{qqq}, and π_{rrr} intersect. Show that, in contrast to the quadratic case, the point C_{pqr} always lies on the plane π_{pqr}.

Problem 2. [15 points]

Let M be the affine map from the plane to the plane whose matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix};$$

so M takes the origin to the origin, but is otherwise arbitrary. If $\xi := (0; x, y)$ is any vector, the ratio

$$\frac{\|M(\xi)\|}{\|\xi\|}$$

of the length of $M(\xi)$ to the length of ξ is the factor by which the affine map M multiplies lengths in the direction ξ. Find formulas, in terms of a, b, c, and d, for the maximum and minimum values of this ratio.

For example, Figure 1 shows what happens when the shearing map S with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is applied to the unit circle in the plane. The unit circle becomes an ellipse whose semi-major and semi-minor axes have lengths $\overline{OA} = (\sqrt{5}+1)/2$ and $\overline{OB} = (\sqrt{5}-1)/2$.

You may find it helpful to express the vector ξ in polar coordinates, say as $\xi = (0; r\cos\theta, r\sin\theta)$, and to recall the trigonometric identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = \cos^2\theta - \sin^2\theta$. You may also want to check your answer by verifying that the product of the maximum and minimum length ratios is $|ad - bc|$, the factor by which M multiplies (unsigned) areas.

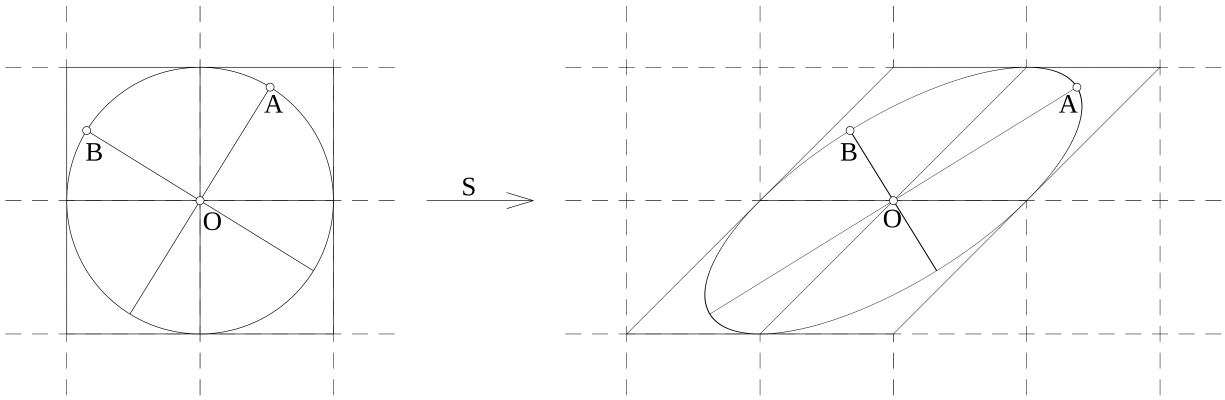

Figure 1: A shearing of the unit circle.

Problem 3. [**10 points**]

Let U and V be (one-sided) projective planes. Find the matrix of the projective map $F: U \to V$ that takes four points to four points as follows:

$$\begin{aligned} F([1;0,0]) &= [1;1,0] \\ F([0;0,1]) &= [1;-1,0] \\ F([1;1,1]) &= [1;0,1] \\ F([4;2,1]) &= [5;3,4]\,. \end{aligned}$$

Recall that the function $B: \mathbf{R} \to U$ defined by $B(t) := [1;t,t^2]$ is a parameterization of the parabola $Y = X^2$ in the plane U. If we transform the parabola B by the projective map F, what curve results? That is, give the implicit equation for the curve traced out in the plane V by the composed function $t \mapsto F(B(t))$.

To what line in V does the projective map F take the line at infinity in U? Give its homogeneous coefficients. What line in U is taken by F to the line at infinity in V? Again, give its homogeneous coefficients.

Problem 4. [**10 points**]

Consider the region R of the two-sided projective plane defined by the conjunction of the following three linear inequalities:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x+y &\geq w. \end{aligned}$$

The region R is a triangle in the two-sided plane. What are the vertices of R? Draw a simple illustration showing what points of the top range and what points of the bottom range lie in R.

Check that the points $A := [1; 1, 1.5]$ and $B := [-1; 2, 1.5]$ lie in R. Draw these points in your diagram.

The line segment AB connecting A to B is the locus of all points of the from $[\lambda A + \mu B]$, for λ and μ positive. Prove that all such points lie in R, and draw a picture of the segment AB in your diagram. Does this segment intersect the line at infinity? If so, at what point?

Problem 5. [15 points]

Let $p := (1 + I)/\sqrt{2}$ and $q := (1 + J)/\sqrt{2}$ denote the unit-norm quaternions used in the example in the reader. Recall that the rotation $M(p)$ is a 90-degree rotation about the X axis, while $M(q)$ is a 90-degree rotation about the Y axis. In the reader, we composed the two rotations $M(p)$ and $M(q)$. Here, we instead investigate the rotation that lies halfway between $M(p)$ and $M(q)$.

The quaternion that lies halfway between p and q is simply

$$\frac{p+q}{2} = \frac{1}{\sqrt{2}} + \frac{I}{2\sqrt{2}} + \frac{J}{2\sqrt{2}}.$$

Calculate the norm $|(p + q)/2|$ of that quaternion, and note that it is not 1. Find a quaternion r that is a scalar multiple of $(p + q)/2$ and that has unit norm, $|r| = 1$, and calculate the rotation matrix $M(r)$. Around what axis does $M(r)$ rotate, and through what angle (say, to the nearest tenth of a degree)?

Find a globe and a piece of string. Say that Leo's Bistro is located at 45 degrees north latitude and 0 degrees longitude, near Bordeaux in France, and that Ravi's Bar and Grill is located at 45 degrees north latitude and 90 degrees west longitude, in the middle of Wisconsin. To the nearest degree, what is the latitude of an airplane that is halfway along a great-circle course from Leo's to Ravi's?

Briefly explain the connection between the two halves of this problem. In particular, which quaternions correspond to which points on the globe?